Homework 6

Goal: To learn to numerically solve ODEs in Matlab using the Runge Kutta-45 integration scheme.

Due: April 23

Problems:

1. ***Harmonic Oscillator.*** Integrate the harmonic oscillator ODE over 4 periods. You will need to rewrite the standard equation of motion $\ddot{x} + \omega_0^2 x = 0$ as two first-order equations:

$$\frac{dx}{dt} = v \qquad\qquad \frac{dv}{dt} = -\omega_0^2 x.$$

   Use $x(0) = 1$ and $v(0) = 0$ as the initial conditions. Set the period of the oscillator to $T = 1$ second.

   (a) Use the built-in Matlab function `ode45()` to perform the integration using the default tolerances. Create three subplots on the same page:

   - subplot #1: Plot the displacement $x(t)$ and the exact solution $x_{exact}(t) = x_0 \cos(\omega_0 t)$.
   - subplot #2: Plot the positional error $\epsilon(t) = x(t) - x_{exact}$.
   - subplot #3: Plot the total energy of the system as a function of time:

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

     Assume the mass is $m = 1$ kg (you will need to calculate the spring constant k from $\omega_0^2 = k/m$).

   The integration routine will adjust the time step automatically, however you can calculate an average step size by dividing the maximum time of the simulation by the number of steps (found by the `length(t)` command).

   Turn in: P1.m, P1_derivs.m, P1.pdf

   (b) Repeat this problem using an Eulerian integration scheme (i.e. just run your Homework 5 problem 3 program). Use the same parameters as in part (a) above. Pick a time step to match the average time step in the Runga Kutta method. Estimate the error in the position and the error in the energy. How do the errors compare? Turn in the plot produced by Euler's method (and label it as such).

   Turn in: P1B.pdf

2. ***Relativistic Spring.*** Newton's second law written in the form:

$$\frac{dp(t)}{dt} = F$$

   turns out to hold for relativistic mechanics, provided that $p(t)$ is given by the relativistic momentum

$$p(t) = \frac{mv(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}}.$$

   To solve for the dynamics of a relativistic mass on a spring, we just substitute $F = -kx$. Plug in the spring force to derive the following equation of motion:

$$\ddot{x} = -\omega_0^2 x \left(1 - \frac{\dot{x}^2}{c^2}\right)^{3/2}.$$

(a) Derive the expression for the equation of motion.

(b) Use the Runge-Kutta `ode45()` scheme to solve the relativistic spring equation. Model a spring with a classical period $T = 1$ s, starting with $x(0) = 1$ m, $v(t) = 0$ m/s over 4 classical periods. Plot the solution and overlay the non-relativistic solution. Do they agree?

Turn in: P2B.m, P2_derivs.m, P2B.pdf

(c) Use a classical calculation to determine how far the spring should be initially stretched so that the mass reaches a maximum velocity of $v = c$ at $x = 0$ (show your work). Use this displacement to simulate the spring over 4 periods. Does your relativistic spring achieve $v = c$? Provide a plot of $x(t)$ and $v(t)$.

Turn in: P2C.m, P2C.pdf

(d) Use the same parameters as in part (c), but this time start the mass off at five times the part (c) value. Again, plot $x(t)$ and $v(t)$ over 4 periods. Describe the motion qualitatively.

Turn in: P2D.m, P2D.pdf