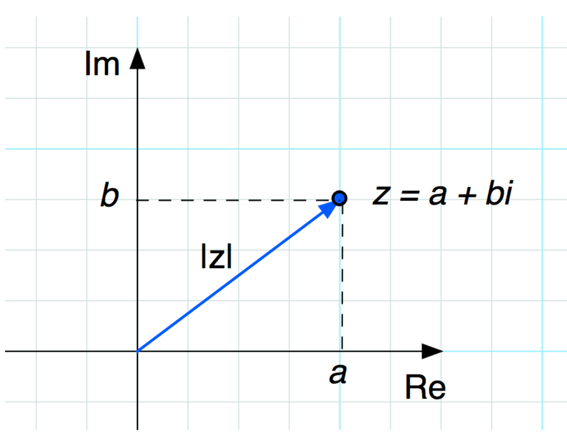
Math Interlude: Complex Numbers and Waves

$$i = \sqrt{-1}$$

Complex Numbers

Complex numbers have real and imaginary parts:

$$z = a + bi \quad \text{where} \quad i = \sqrt{-1}$$



example: $z = 4 + 3i$

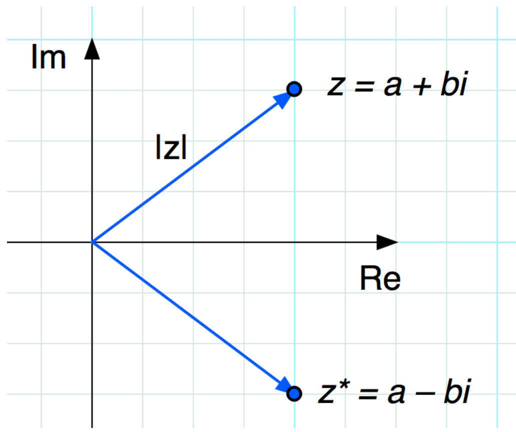
$\mathrm{Re}(z) = 4$

$\mathrm{Im}(z) = 3$

Complex Conjugate

Given a complex number $z = a + bi$

The complex conjugate is defined as $z^* = a - bi$



Problem: Find the complex conjugate of the following:

$$z = \frac{1}{1 + 2i}$$

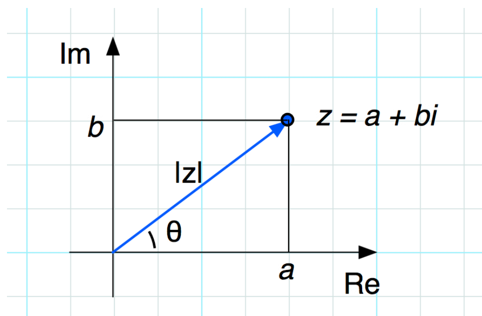
Polar Form of Complex Number

Magnitude (modulus): $|z|^2 = z^*z = (a - bi)(a + bi)$

$$|z|^2 = a^2 + b^2$$

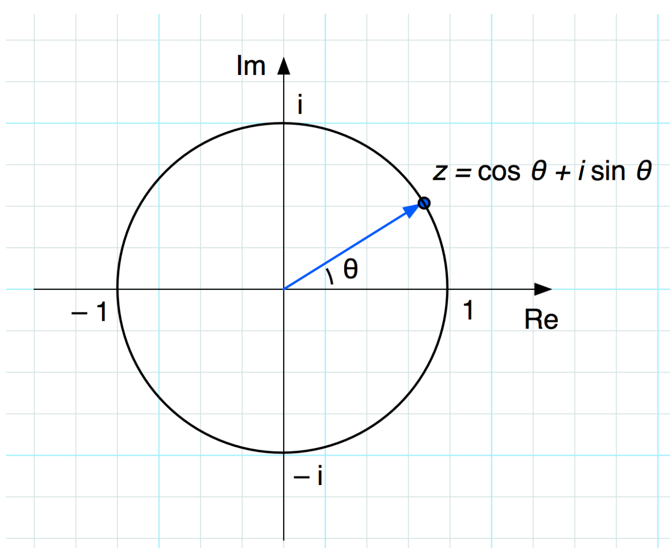
Angle from real axis = θ: $a = |z|\cos\theta$ $\quad z = |z|\cos\theta + i|z|\sin\theta$

$$b = |z|\sin\theta$$



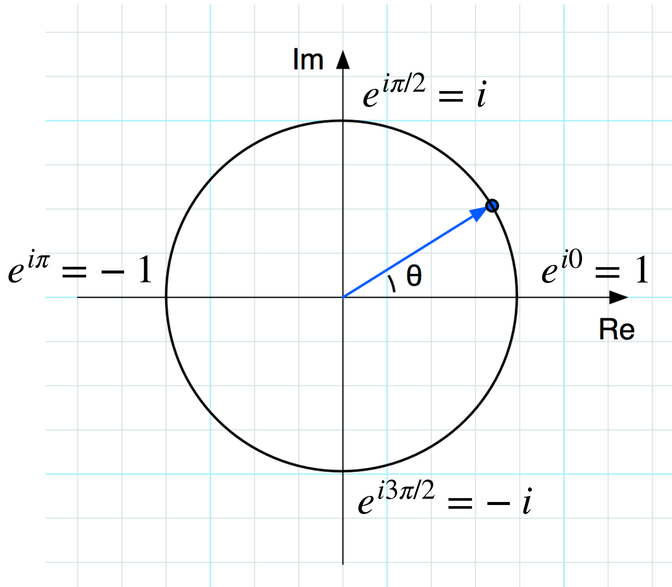
Unit Circle

Set magnitude to unity: $z = \cos\theta + i\sin\theta$



Complex Exponential and Euler's Formula

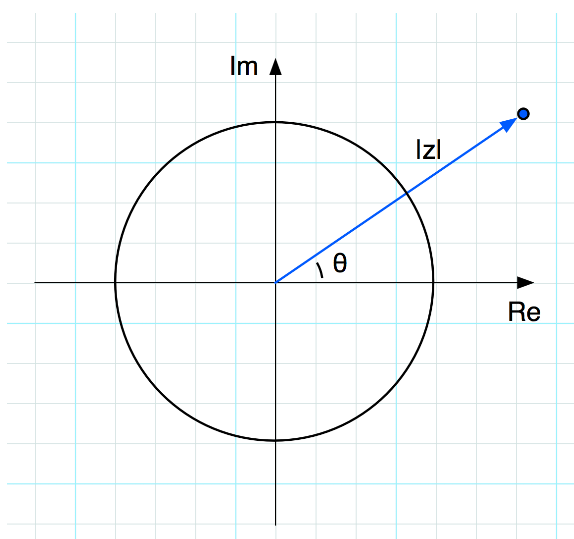
$$e^{i\theta} = \cos\theta + i\sin\theta$$



$$e^{i0} = 1$$
$$e^{i\pi/2} = i$$
$$e^{i\pi} = -1$$
$$e^{i3\pi/2} = -i$$

Complex Exponential Notation

$$z = |z|e^{i\theta} = |z|\cos\theta + i|z|\sin\theta$$



Problem: Write the following complex number in polar form

$$z = 1 + \sqrt{3}i$$

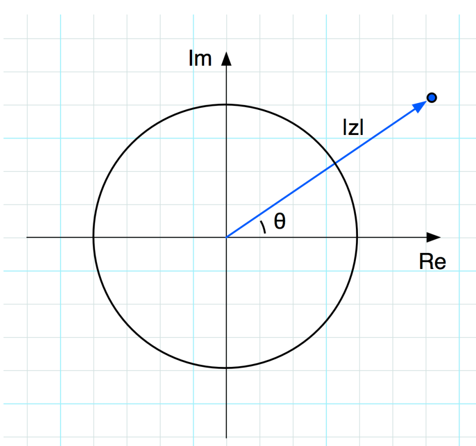
Problem: Write the following polar complex number in a+bi form

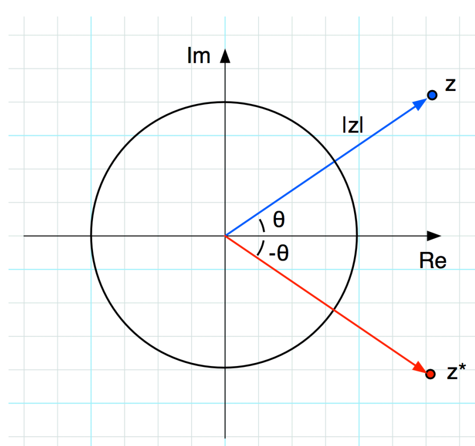
$$z = \frac{1}{2}e^{i3\pi/4}$$

Complex Conjugate of a Complex Exponential

$$z = |z|e^{i\theta} = |z|\cos\theta + i|z|\sin\theta$$

$$z^* = |z|e^{-i\theta} = |z|\cos\theta - i|z|\sin\theta$$





Useful Identities

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

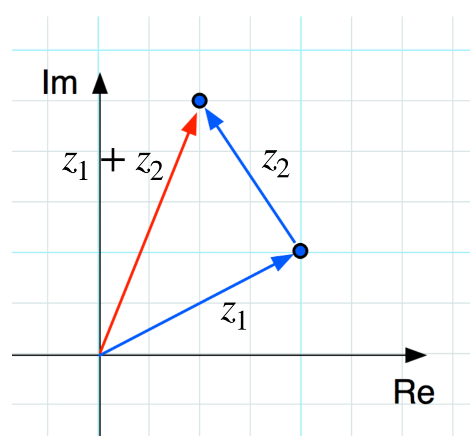
$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

Addition of Complex Numbers

It is easiest to add complex numbers when they are in a+bi form.
They add like vectors.

$$z_1 = a_1 + b_1 i$$
$$z_2 = a_2 + b_2 i$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$



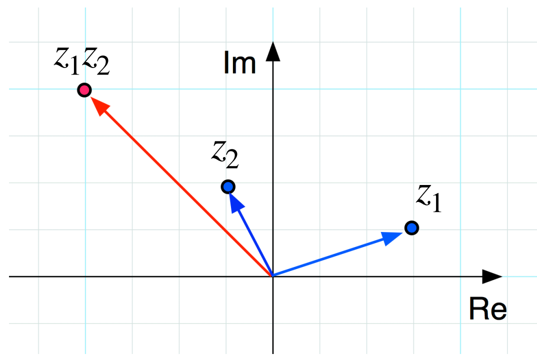
Multiplication of Complex Numbers

What happens when we multiply two complex numbers?
Use FOIL to multiply the following:

$z_1 = 3 + i$

$z_2 = -1 + 2i$

$z_1 z_2 =$



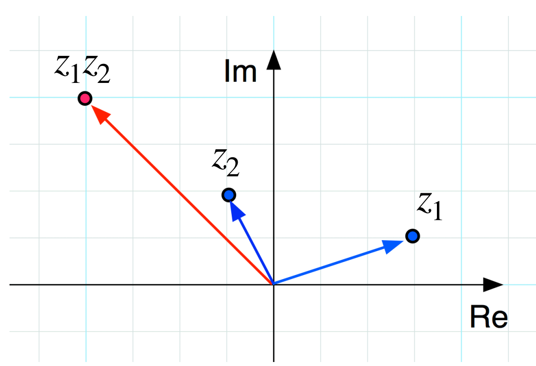
Multiplication of Complex Numbers

What happens when we multiply two complex numbers?
Use FOIL to multiply the following:

$z_1 = 3 + i$

$z_2 = -1 + 2i$

$$z_1 z_2 = (1 + 2i)(2 + 3i)$$
$$= -5 + 5i$$



Multiplication of Complex Numbers Using Complex Exponents

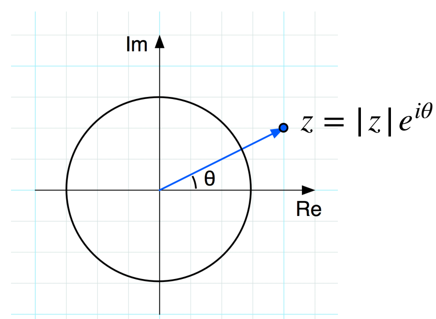
Find the product $z_3 = z_1 z_2$ where $z_1 = |z_1| e^{i\theta_1}$ and $z_2 = |z_2| e^{i\theta_2}$

$$z_1 z_2 = \left(|z_1| e^{i\theta_1}\right)\left(|z_2| e^{i\theta_2}\right)$$
$$= |z_1||z_2| e^{i\theta_1} e^{i\theta_2}$$
$$= |z_1||z_2| e^{i(\theta_1+\theta_2)}$$

The result:
(1) moduli are multiplied: $|z_3| = |z_1||z_2|$
(2) angles are added: $\theta_3 = \theta_1 + \theta_2$

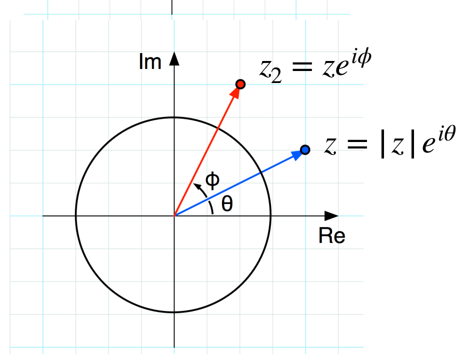
Multiplication of Complex Number z by $e^{i\varphi}$ rotates it in the complex plane by angle φ

Initial complex number: $z = |z| e^{i\theta}$



Multiply z by $e^{i\phi}$: $z_2 = z e^{i\phi}$

$$z_2 = |z| e^{i\theta} e^{i\phi}$$
$$z_2 = |z| e^{i(\theta+\phi)}$$



What is the effect of dividing two complex numbers?

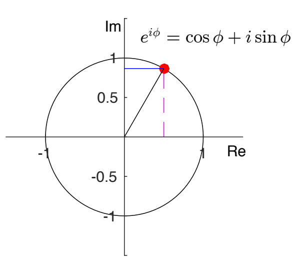
$$z = \frac{|z_2| e^{i\theta_2}}{|z_1| e^{i\theta_1}} = ?$$

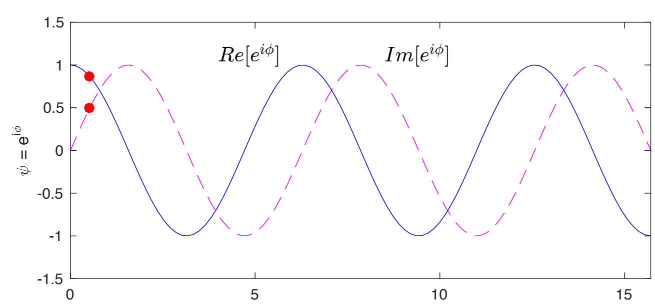
Simple Harmonic Motion

Let the complex exponential $e^{i\phi}$ depend on time with $\phi = \omega t$

$$z(t) = e^{i\omega t}$$

$$z(t) = \underbrace{\cos \omega t}_{\text{Re}} + i \underbrace{\sin \omega t}_{\text{Re}}$$

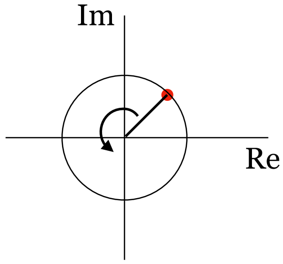




Simple Harmonic Motion

counterclockwise rotation

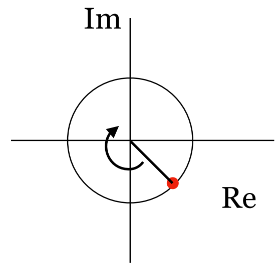
$z(t) = e^{i\omega t}$



$Re[e^{i\omega t}] = \cos \omega t$

clockwise rotation

$z(t) = e^{-i\omega t}$



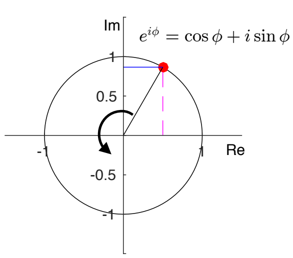
$Re[e^{-i\omega t}] = \cos \omega t$

$e^{i\omega t}$ vs. $\cos \omega t$

$z(t) = e^{i\omega t}$

$z(t) = \cos \omega t + i \sin \omega t$

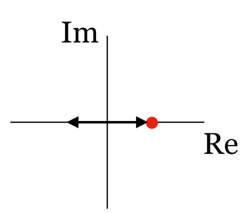
$|z(t)|^2 = 1$



$y(t) = \cos \omega t$

$y(t) = Re[z(t)]$

$y(t)^2 = \cos^2 \omega t$



Harmonic Traveling Wave

Real-valued function: $\eta(x,t) = A\cos(kx - \omega t)$ (rightward-moving)

Complex notation: $\psi(x,t) = Ae^{i(kx-\omega t)}$

$$Re[\psi(x,t)] = A\cos(kx - \omega t)$$

$$Im[\psi(x,t)] = A\sin(kx - \omega t)$$

$$|\psi(x,t)|^2 = A^2$$

Standing Wave

Real-valued function: $\eta(x,t) = A\cos(kx + \phi_1)\,\cos(\omega t + \phi_2)$

where ϕ_1 and ϕ_2 are constant phase factors.

Complex notation: $\psi(x,t) = A\cos(kx)\,e^{i\omega t}$

$$Re[\psi(x,t)] =$$

$$Im[\psi(x,t)] =$$

$$|\psi(x,t)|^2 =$$

Problem: Use complex exponentials to show that a standing wave can be created by adding two oppositely moving traveling waves.