Lecture 4. Relativistic Momentum and Energy

Contents

Summary. In the last set of lecture notes, we saw that many of our ideas from classical physics must be modified to satisfy Einstein's postulates. We saw that space and time "warp" into each other leading to the phenomena of time dilation and length contraction. Even simple notions like the addition of velocities must be modified to be compatible with relativity. The last set of lecture notes dealt with relativistic **kinematics** (i.e. the description of motion). In this lecture, we will explore relativistic **dynamics** to see how the notions of force, work, energy and momentum translate to relativity.

4.1 Relativistic Momentum

Let's imagine that two famous physicists (Fabiola Gianotti and Rosalyn Sussman Yalow) each have a ball. At first they are at rest with respect to each other. They each throw their ball with speed v_0 so the balls collide half way between them. We'll assume a completely elastic collision so the balls bounce directly back. After the collision, each ball will reverse its velocity, the total momentum will remain zero and momentum will be conserved. This experiment is simple but offers little insight into the nature of relativistic momentum.

Let's now consider the same collision, except that Rosalyn and Fabiola will be in relative motion to each other (see Figure 1). Define S as Rosalyn's rest frame and S' as Fabiola's rest frame. Let's assume Fabiola moves with velocity u in the $+x$ direction relative to Rosalyn. Fabiola and Rosalyn each throw their ball with speed v_0 directed along the $-y'$ and $+y$ axes respectively. Let's call the velocity of Rosalyn's ball $\vec{\mathbf{v}}$ and the velocity of Fabiola's ball $\vec{\mathbf{w}}$ as measured in Rosalyn's frame of reference (S). In Gabiola's frame (S'), the velocities are $\vec{\mathbf{v}}'$ and $\vec{\mathbf{w}}'$ respectively. The velocity of Rosalyn's ball in her frame and the velocity of Fabiola's ball in her frame are easy to write down:

$$\vec{\mathbf{v}} = (0, v_0) \qquad \vec{\mathbf{w}}' = (0, -v_0).$$

Let's now calculate the velocity of Fabiola's ball in Rosalyn's frame. The x component is simply $w_x = u$ since $w'_x = 0$ and the relative velocity of the reference frames is u. We use the the velocity

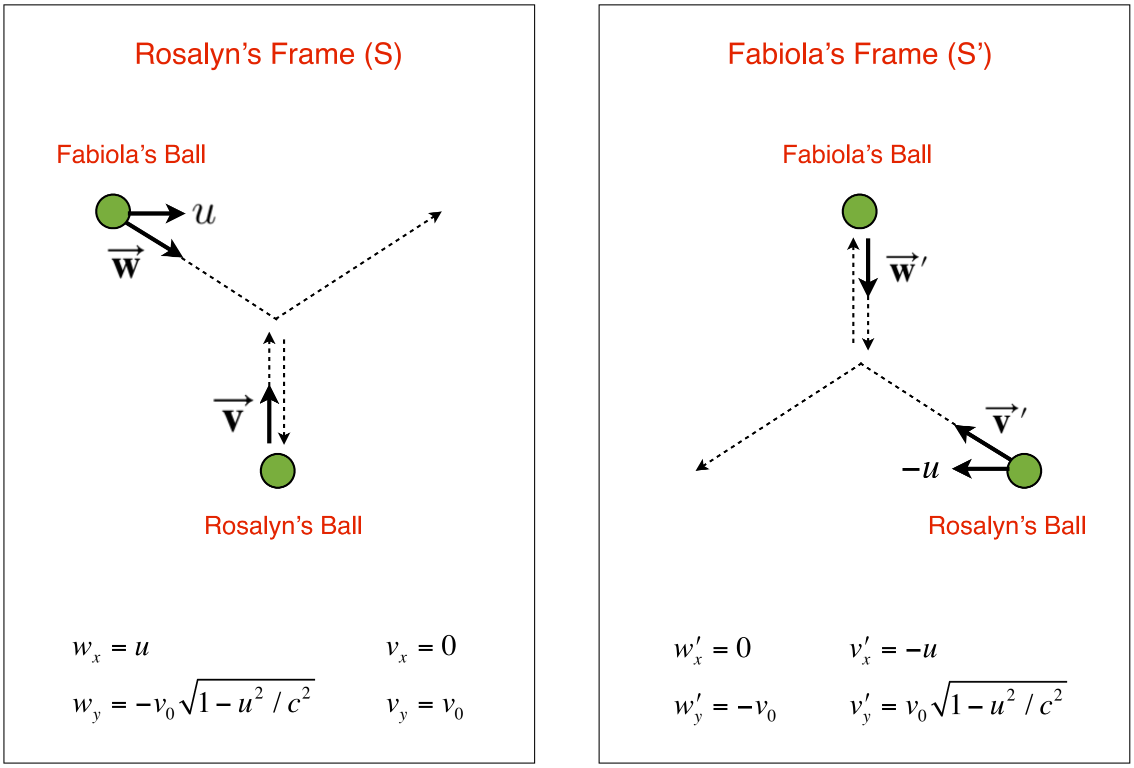

Figure 1: Geometry of an elastic collision. Rosalyn and Rabiola are in relative motion along the x axis.

addition formula to calculate w_y:

$$w_x = u$$
$$w_y = \frac{w'_y\sqrt{1-u^2/c^2}}{(1+uw'_x/c^2)} = \frac{-v_0\sqrt{1-u^2/c^2}}{(1+u\cdot 0/c^2)} = -v_0\sqrt{1-u^2/c^2}$$

Notice that $|w_y| < v_0$, i.e. Rosalyn sees Fabiola throw her ball *slower* than she throws her own ballv_0 by a factor of $1/\gamma = \sqrt{1-u^2/c^2}$. Similarly, Fabiola would see Rosalyn throw her ball slower than she throws her on ball. This effect is due to time dilation between the moving frames.

The above calculation means that the classical momentum of Rosalyn's ball is greater the classical momentum of Fabiola's ball (as measured in Rosalyn's frame, i.e. $mv_y > m|w_y|$. Had we done the calculation in Fabiola's frame, we would have found the opposite, that $m|w'_y| > mv'_y$. But according to Einstein's first postulate, the relative momentum of the two balls cannot depend on the reference frame chosen, so our classical definition of momentum $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$ cannot apply to relativity.

In order for the momenta of the two balls to be consistent, we must define the **relativistic momentum** as

$$\vec{\mathbf{p}} = \gamma(v)m\vec{\mathbf{v}} \tag{4.10}$$

where we introduce

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

We should point out a few things:

- This gamma factor depends on the speed of the *object* $v = |\vec{v}|$, not the relative velocity between our observers u. We add the functional depend $\gamma(v)$ to distinguish this gamma factor from γ (which depends on u not v). Don't confuse them! If it helps, you can start writing the normal gamma factor as $\gamma(u)$ to tell them apart.

- The $\gamma(v)$ term always depends on the total speed of the object, even if only a single momentum component is being calculated. For example, if you want to find the x component of the momentum you could write

  $$p_x = \gamma(v) m v_x,$$

  where the gamma factor depends on $v = \sqrt{v_x^x + v_y^2 + v_z^2}$.

- In the limit that $v \ll c$, $\gamma(v) \approx 1$ and we recover the classical limit that $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$. As the velocity approaches the speed of light, the relativistic momentum diverges to infinity.

Let's test our our new definition of momentum and see if Rosalyn measures the same y-component of momentum for each ball. In other words, we want to see if $\gamma(w) m w_y = -\gamma(v) m v_y$. Here's the momentum of Rosalyn's ball in her reference frame:

$$\begin{aligned} p_y(\text{Rosalyn}) = \gamma(v) m v_y &= \frac{m v_y}{\sqrt{1 - v^2/c^2}}. \\ &= \frac{m v_0}{\sqrt{1 - v_0^2/c^2}}, \end{aligned}$$

where we used $v_y = v_0 = v$ (why is this true?). And here's the momentum of Fabiola's ball in Rosalyn's frame:

$$p_y(\text{Fabiola}) = \gamma(w) m w_y = \frac{m w_y}{\sqrt{1 - w^2/c^2}}.$$

We calculate the speed w from its components

$$\begin{aligned} w^2 &= w_x^2 + w_y^2 \\ &= u^2 + v_0^2(1 - u^2/c^2) \end{aligned}$$

and substitute in for w_y in the numerator to give

$$\begin{aligned} \gamma(w) m w_y &= \frac{-m v_0 \sqrt{1 - u^2/c^2}}{\sqrt{1 - (u^2 + v_0^2(1 - u^2/c^2)/c^2}} \\ &= \frac{-m v_0}{\sqrt{1 - v_0^2/c^2}}. \end{aligned}$$

We see that

$$p_y(\text{Fabiola}) = -p_y(\text{Rosalyn})$$

so the momenta sum to zero as they should.

Example 1. Find the fractional change in momentum when a particle doubles its speed from (a) $0.1c$ **to** $0.2c$**. (b) Repeat for a change in speed from** $0.4c$ **to** $0.8c$**.**

Solution. We consider two velocities v_1 and v_2.

$$\begin{aligned}\frac{p_2}{p_1} &= \frac{\gamma(v_2)mv_2}{\gamma(v_1)mv_1}\\ &= \frac{mv_2/\sqrt{1-v_2^2/c^2}}{mv_1/\sqrt{1-v_1^2/c^2}}\\ &= \frac{v_2}{v_1}\sqrt{\frac{1-v_1^2/c^2}{1-v_2^2/c^2}}\end{aligned}$$

(a) When $v_1 = 0.1c$ and $v_2 = 0.2c$.

$$\frac{p_2}{p_1} = 2.03.$$

This result only differs from the classical prediction (which is $p_2/p_1 = 2$) by less than 2%.

(b) When $v_1 = 0.4c$ and $v_2 = 0.8c$.

$$\frac{p_2}{p_1} = 3.06.$$

In this case, the ratio is more than 50% greater than the classical prediction. As one approaches the speed of light, the relativistic momentum diverges more and more from the classical prediction.

4.2 Newton's Second Law

In classical physics, Newton's second law may be written in two ways:

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} \qquad \text{(classical - NOT relativistic)}$$

or

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \qquad \text{(both classical and relativistic).} \tag{4.11}$$

Interestingly, the famous equation $F = ma$ is *not* consistent with relativity, while the second equation (Equation 2.11) is. We will not *prove* that Equation 2.11 is relativistically valid, rather we ask you to trust us when we say it is. However, we can show that the classical equation $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ is *not* consistent with Equation 2.11 in relativity by simply plugging in the expression for momentum

(Equation 2.10) into Equation 2.11. To keep things simple, we'll stick to 1D:

$$\begin{aligned}
F &= \frac{dp}{dt} \\
&= \frac{d}{dt}\left[\gamma(v)mv\right] \\
&= \frac{d}{dt}\left[\frac{mv}{\sqrt{1-v^2/c^2}}\right]
\end{aligned}$$

After differentiating and simplifying, we can show that

$$F = \gamma^3(v)ma. \tag{4.12}$$

Equation 4.12 is the relativistic version of $F = ma$. Note the presence of the $\gamma^3(v)$ factor.

- **Problem 1 - Fill in the missing steps to derive $F = \gamma^3(v)ma$.**

4.3 Work

The classical definition of work W translates over to relativity without modification:

$$W = \int_0^x F(x)dx \qquad \text{(both classical and relativistic).} \tag{4.13}$$

4.4 Relativistic Kinetic Energy

Our classical definition of kinetic energy does not work in relativity

$$K = \frac{1}{2}mv^2 \qquad \text{(classical).}$$

To derive a relativistically valid expression for kinetic energy, we use the work-kinetic energy theorem, which remains valid in relativity:

$$\Delta K = K_{final} - K_{init} = W = \int_0^x F(x)dx \qquad \text{(both classical and relativistic).} \tag{4.14}$$

We use Equation 4.11 and "play around" with the differentials:

$$\begin{aligned}
\Delta K &= \int_0^x F(x)dx \\
&= \int_0^x \frac{d}{dt}(\gamma(v)mv)dx \\
&= \int_0^x d(\gamma(v)mv)\frac{dx}{dt} \\
&= \int_0^x vd(\gamma(v)mv).
\end{aligned}$$

Next, make the substitution $q = \gamma mv$ and integrate by parts $\int vdq = vq - \int qdv$. After simplifying and setting $K = \Delta K$ (since $K_{init} = 0$), one obtains

$$K = (\gamma - 1)mc^2. \tag{4.15}$$

Let's evaluate the kinetic energy in two limits:

- When $v = 0$, $\gamma = 1$ so $K = (1-1)mc^2 = 0$. Thus, we see that the kinetic energy is zero when the object is at rest in the observer's reference frame.
- In the limit that $v \to c$, $\gamma \to \infty$ so the kinetic energy diverges to infinity. This says that if an object could be accelerated to the speed of light, it would carry an infinite amount of kinetic energy. Because this would require an infinite amount of work (work-kinetic energy theorem), it is impossible to ever accelerate an massive object to light speed. Thus, the speed of light serves as a universal speed limit.

- **Problem 2 - Fill in the missing steps to derive $K = (\gamma - 1)mc^2$.**
- **Problem 3 - Show that the relativistic kinetic energy reduces to $K = \frac{1}{2}mv^2$ in the limit that $v << c$.**

4.5 Total Energy and Rest Energy

Equation 4.15 may be written as

$$K = \gamma mc^2 - mc^2.$$

The two terms on the right-hand-side of the equation have special meaning. The first term is defined as the **total energy** E of the object

$$E = \gamma mc^2 \tag{4.16}$$

and the second term is defined as the **rest energy** E_0, which leads to perhaps the most famous equation in all of physics

$$E_0 = mc^2. \tag{4.17}$$

After substituting these definitions and rearranging, we see that the total energy equals the kinetic energy plus the rest energy of the object:

$$E = K + E_0. \tag{4.18}$$

When the velocity of the object is zero, its kinetic energy is zero (just as in classical physics). However, we see that the object still has some energy E_0 even when it is at rest. Einstein interpreted this result to mean that mass and energy are fundamentally equivalent: mass may be converted to energy and energy may be converted to mass. The rest energy $E_0 = mc^2$ may be thought of as the energy "stored" in mass m. This equivalency also lets us write the mass of objects in units of energy / c^2. See the table below for some example particle masses.

Particle	Symbol	mass (kg)	mass (a.m.u.)	mass (eV/c^2)
electron	e^-	$9.1093829 \times 10^{-31}$ kg	5.485799 u	0.5109989 MeV/c^2
proton	p	$1.67262178 \times 10^{-27}$ kg	1.0072765 u	938.27205 MeV/c^2
neutron	n	$1.67492735 \times 10^{-27}$ kg	1.0086649 u	939.56538 MeV/c^2

4.6 Energy-Momentum Relationship

In classical physics we know that the kinetic energy and momentum are related by

$$K = \frac{p^2}{2m} \qquad \text{(classical)}.$$

This relation does *not* hold in relativity. However, we can derive a relationship between the total energy E and momentum p that does work in relativity:

$$E^2 = (pc)^2 + (mc^2)^2 \qquad \text{(relativistic)}. \tag{4.19}$$

- **Problem 4 - Starting with the definitions of the total energy E and momentum p derive $E^2 = (pc)^2 + (mc^2)^2$.**

4.7 Units

In high energy physics and particle physics, energy is often measured in units of electron volts (1 eV = 1.60×10^{-19} J), with appropriate prefixes (1 keV = 10^3 eV, 1 MeV = 10^6 eV, 1 GeV = 10^9 eV, and 1 TeV = 10^{12} eV).

Because of Einstein's statement of mass-energy equivalence $E_0 = mc^2$, mass may be written in terms of energy. Thus eV/c^2 is a unit of mass.

Similarly, we see that eV/c is a unit of momentum.

Example 2. Find the velocity of an electron that has a kinetic energy of 2 MeV.

Solution. We start with the equation for the kinetic energy (Equation 4.15) and solve for the gamma term:

$$\begin{aligned}\gamma(v) &= 1 + \frac{K}{mc^2}. \\ &= 1 + \frac{2 \text{ MeV}}{0.511 \text{ MeV}} \\ &= 4.91.\end{aligned}$$

Now find the velocity based on this gamma term:

$$v = \sqrt{1 - \frac{1}{\gamma^2}}.$$

Plugging in for γ gives $u = 0.892c$

4.8 Massless Particles

Equation 4.19 gives an interesting result when the mass m of the particle is zero. This equation tells us

$$E = pc \tag{4.20}$$

when $m = 0$. In other words, massless particles with energy E carry momentum $p = E/c$. The photon is the best-known example of a massless particle.

4.9 Highly Relativistic Particles

We define a highly relativistic particle as those whose kinetic energy is much greater than its rest energy, i.e.

$$K >> mc^2.$$

For such particles, $K + mc^2 \approx K$, so $E = K + mc^2 \approx K$. Thus the total energy is also much larger than the rest energy

$$E >> mc^2.$$

This means that the mc^2 term in Equation 2.19 is small compared with E, so Equation 4.19 simplifies to

$$E \approx pc. \tag{4.21}$$

To a first approximation the energy-momentum relationship for highly relativistic particles (Equation 4.21) identical to that for a massless particle (Equation 4.20).

- **Problem 5 - Find the velocity of a highly relativistic proton (in terms of c) such that $K = 10mc^2$. Compare the true momentum of the proton calculated from $E^2 = (pc)^2 + (mc^2)^2$ with the approximation $p = E/c$. What fractional error in p does this correspond to?**
