# Lecture 2. Postulates of Special Relativity

## Contents




**Summary.** Einstein developed the special theory of relativity as a means of solving a number of problems he saw in Maxwell's equations of electrodynamics. In this lecture, we review Galilean relativity, discuss the deficiencies Einstein saw in Maxwell's equations, and present the Lorentz transformations that replace Galilean transformations under special relativity.


### 2.1 Frames of Reference

A **frame of reference** is a coordinate system used to make observations of the world. We can imagine attaching a reference frame to the classroom, to someone riding a bicycle or to an electron traveling through space (see Figure 1, for example). An **inertial reference frame** is a reference frame in which the law of inertia is valid, i.e. it is a frame that has zero acceleration. For most experiments, we can consider the classroom to be an inertial reference frame. However, because the classroom is attached to the Earth (which is spinning on its axis and orbiting around the Sun) the classroom is actually accelerating slightly and is thus not a *perfect* inertial reference frame.

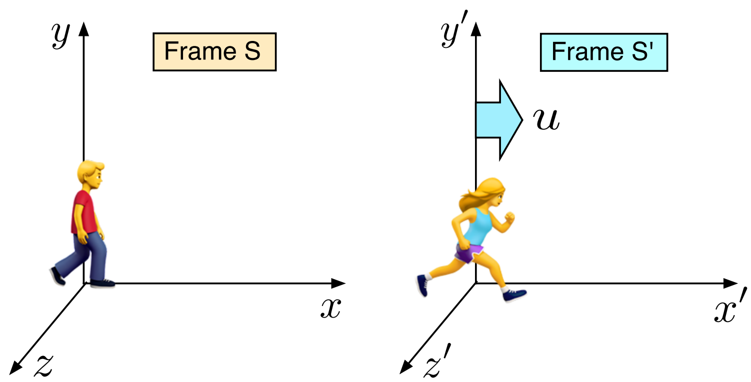


Figure 1: Two inertial reference frames $S$ and $S'$. We assume frame $S'$ is moving with velocity $u$ with respect to frame $S$

## 2.2 Galilean Relativity

Galilean Relativity is based on the common-sense idea that space and time are independent concepts, i.e. that the flow of time and extent of space do not depend on one's position or motion. Galilean relativity works just fine with Newton's Laws and our every-day experience.

Let's consider two inertial reference frames, one attached to Albert and and one attached to Mileva. We will place Albert at the origin of his reference frame, and Mileva at the origin of hers (see Figure 1). As Albert walks along, his reference frame moves with him. The same is true for Mileva. *One is always at rest with respect to one's own reference frame.* We'll call Albert observer $O$, his reference frame $S$, and the coordinates of an object measured in his frame $(x, y, z)$. Similarly, Mileva will be observer $O'$, her frame is $S'$, and the coordinates defining an object in her frame will be $(x', y', z')$. We'll also give each observer a stopwatch and call the reading on Albert's watch $t$ and the reading on Mileva's watch $t'$.

Let's assume Mileva runs past Albert along Albert's $x$ axis. We'll call her velocity relative to him $u$. (What is Albert's velocity relative to Mileva?) We will also assume both Albert and Mileva start their stop watches the instant Mileva passes Albert. We set up the coordinate systems so that Mileva's $x'$ axis slides along Albert's $x$ axis. When $t = 0$, $t' = 0$ and the $S$ and $S'$ frames will be coincident. Thus, at some time $t$ later, the origin of Mileva's frame will be at $x = ut$ in Albert's frame (see Figure 2).

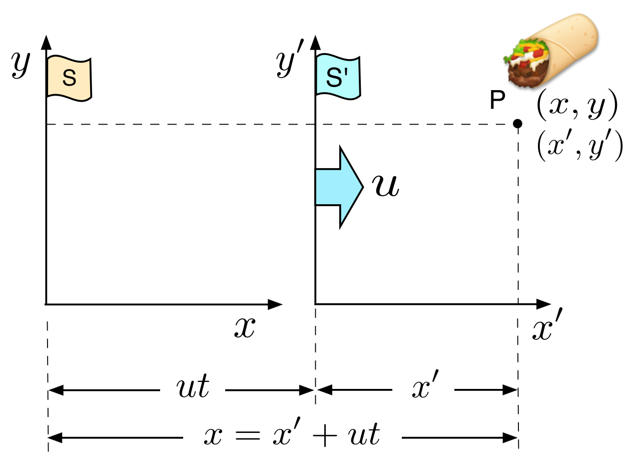


Figure 2: Position measurement of an object (a burrito) made from two coordinate frames $S$ and $S'$.

Now, let's introduce a third object, say a burrito (again, see Figure 2). Albert will measure the coordinates of the burrito as $(x, y, z)$ and Mileva will measure its coordiantes as $(x', y', z')$. Because of the way we defined the coordinates, $y' = y$ and $z' = z$ and $x = x' + ut$. Do these equations make sense? If not, study Figure 2. Collecting these equations together and solving for $x'$, we arrive at the **Galilean transformation equations**:

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{2.1}$$

Notice that under this transformation, time flows at the same rate for each observer, i.e. $t = t'$,

Differentiating the above relationships and defining $v_x = dx/dt$, $v'_x = dx'/dt$, $v_y = dy/dt$, etc. gives a simple rule for adding velocities

$$\begin{aligned} v'_x &= v_x - u \\ v'_y &= v_y \\ v'_z &= v_z. \end{aligned} \tag{2.2}$$

Returning to Figure 2, we can interpret $v_x$ as the velocity of the burrito as measured by Albert and $v'_x$ is the velocity of the burrito as measured by Mileva. Again, $u$ is the velocity of Mileva as measured by Albert.

In general, if the $S'$ frame is moving with an arbitrary velocity $\vec{u}$ with respect to the $S$ frame, then the relationship between the velocity $\vec{v}$ of an object measured in the $S$ frame and the velocity $\vec{v}'$ measured in the $S'$ frame is

$$\vec{v}' = \vec{v} - \vec{u}. \tag{2.3}$$

These transformation equations define what we mean by **Galilean Relativity**.

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**Example 1. Relative Velocities.**

Suppose Albert is driving down the freeway and Mileva passes him. Albert estimates that Mileva it traveling 20 mph faster that he is. A Porsche passes them both. Albert guesses that the Porsche was going about 50 mph faster than him. How fast is the Porsche moving relative to Mileva?

**Solution.** Let's make the following definitions:

- $u = 20$ mph = the velocity of Mileva relative to Albert
- $v_x = 50$ mph = the velocity of the Porsche relative to Albert
- $v'_x$ = the velocity of the Porsche relative to Mileva

We use the Galilean velocity transformation equations to solve for $v'_x$:

$$\begin{aligned} v'_x &= v_x - u \\ &= 50 \text{ mph} - 20\text{mph} = 30 \text{ mph}. \end{aligned}$$

This result should be fairly intuitive. Another way of saying it is that the velocity of the Porsche as measured by Albert $v_x$ equals the velocity of the porsche as measured by Mileva $v'_x$ plus the velocity of Mileva relative to Albert $u$, i.e.

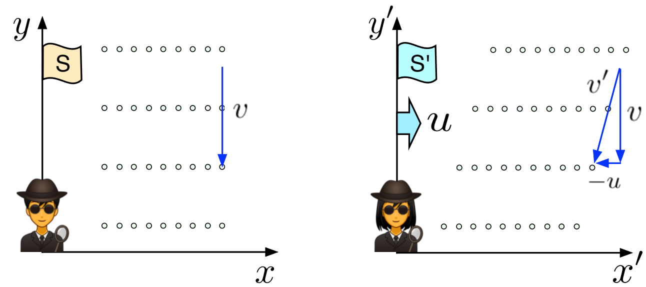
$$v_x = v'_x + u.$$

or

$$v_x = 30\text{ mph} + 20\text{mph} = 50\text{ mph}.$$

**Example 2. Two Observers in a Rain Storm.**

The figure at right shows two observers, Albert and Mileva, in a rain storm. In Albert's $S$ frame, the rain falls straight down with a velocity $\vec{v}$. Mileva moves to the right with velocity $\vec{u}$ with respect to Albert. Find the velocity of the rain in her reference frame.




**Solution.** Let's define the *speed* of the rain as $v = |\vec{v}|$ and Mileva's *speed* with respect to Albert as $u = |\vec{u}|$. Conceptually, we know that because Mileva's motion is along the $x$ axis, she will measure the same downward velocity for the rain as does Albert. However, since she is moving to the right with speed $u$ respect to Albert, then the rain will have a $-u$ component (to the left). Using equation 2.3, we write the velocity of the rain in Mileva's frame as:

$$\vec{v}' = \vec{v} - \vec{u},$$

or

$$v' = -u\,\hat{i} - v\hat{j}.$$

Since the rain's velocity is down and to the left, it will appear to come from a point up and to the right from Mileva's perspective. Thus, if she has an umbrella, she should hold it slightly in front of her to keep from getting wet. The angle by which she must tilt her umbrella is given by

$$\tan\alpha = \frac{-v'_x}{-v'_y} = \frac{u}{v}.$$

Notice that the *speed* of the rain is higher for Mileva than for Albert. In Mileva's frame, the speed is

$$v' = \sqrt{u^2 + v^2}$$

whereas for Albert it is just $v$. Does this make sense? Because Mileva is running into the raindrops, they will hit her faster and harder. For example, if she is running to the right with the same speed that the raindrops are falling, they will hit her with a speed of $\sqrt{2}v$, and they will appear to fall at a 45° angle.

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## 2.3 What's Wrong with Galilean Relativity?

The transformation equations defining Galilean relativity work great for everyday life and are a fundamental part of Classical Physics. However, toward the end of the $19^{th}$ Century, two problems or "mysteries" surfaced that eventually led Einstien to abandon Galilean relativity and replace it with his own theory of special relativity. Both problems are related to Maxwell's theory of electromagnetism. The first had to do with the speed of light (i.e. electromagnetic waves). The second had to do with Faraday's law.

## 2.4 The Problem with the Ether

In the last lecture, we were left with two seemingly contradictory results: (1) Maxwell's equations predicted that the speed of light was a constant (presumably with respect to some hypothetical ether) and (2) Michelson and Morley's experiment showed no evidence of the ether. We are left with two possibilities that will be discussed in turn.

The first possibility was investigated by Lorentz. He assumed that the ether really did exist, but that its affect on Michelson and Morley's experiment was somehow "hidden." Lorentz postulated that perhaps when objects move relative to the ether, they contract just enough so that the light-travel time along the two paths of Michelson's interferometer will remain constant. He did the math, and came up with the Lorentz transformations that we will discuss later in this lecture. Let us be clear: Lorentz couldn't let go of the idea that the ether was real so he figured out a way to keep the ether, but at the expense of introducing a new physical effect, i.e. length contraction.

The second possibility was investigated by Albert Einstein. He did away with the ether all together. But without the ether, what does it mean for the speed of light to be constant? What is it constant relative to? Einstein postulated that the speed of light is constant with respect to *everyone* and *everything* in the universe. Amazingly, if one accepts that assumption, then space and time are related in exactly the way Lorentz postulated[1].

The bottom line is that Einstein solved the first "mystery" discussed above by postulating that the speed of light is constant for all observers. We'll return to this in the next lecture to see what implications this has.

## 2.5 The Problem with Faraday's Law

Einstein was also troubled by a second apparent inconsistency in Maxwell's equations. It had to do with Faraday's Law. Remember that Faraday's law describes the creation of an electric field from a time-varying magnetic field:

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}.$$

Consider a magnet and a conducting ring in relative motion as shown in Figure 3. From the ring's perspective, the magnet is in motion. As the magnet approaches the ring, the magnetic flux through the ring will change, inducing an electric field. This electric field will accelerate electrons in the ring creating an induced current. However, from the perspective of the magnet, the ring is the object in motion. In this case, the magnet is stationary, so no electric field will be produced. As

[1]We note that Einstein interprets the velocity $u$ in the Lorentz transformations as the relative velocity between two observers, wheras Lorentz interpreted it to be the velocity between an object and the ether. But we are getting ahead of ourselves here.

the ring moves to the right with velocity $v$ it will carry its electrons along so they will also have a velocity $v$. These moving electrons will feel a magnetic force $q\vec{v} \times \vec{B}$ that will accelerate them to generate the induced current. In both cases the relative motion between the magnet and the ring creates an induced current in the ring. However in one frame of reference the current is due to an *electric* field and in the other frame of reference it is due to a *magnetic* field. How can both of these perspectives be correct?

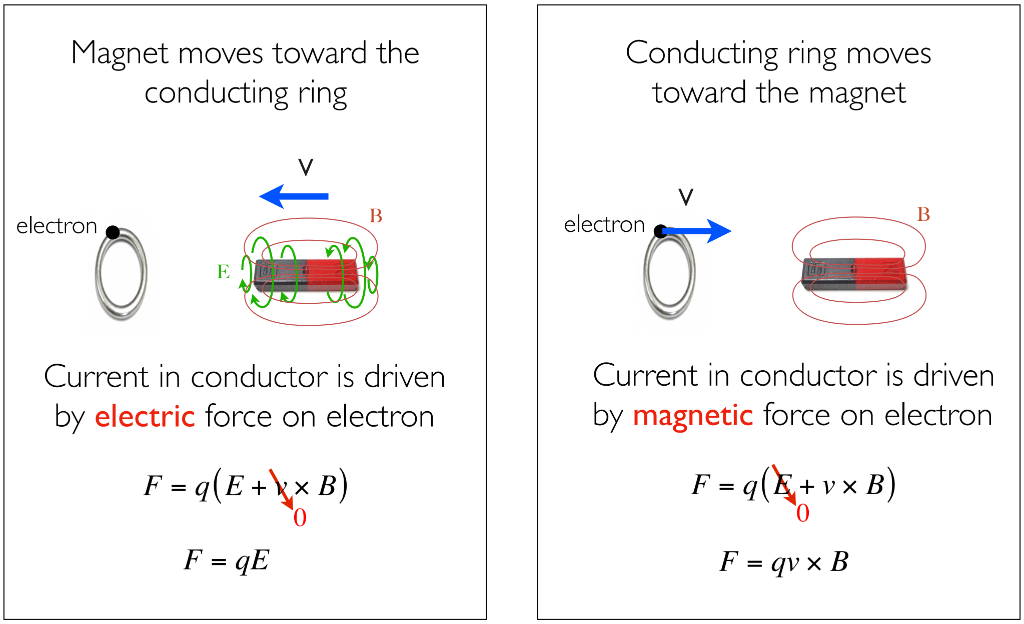


Figure 3: Asymmetry in Faraday's Law. The two panels show the same relative motion between a magnet and a conducting ring. However, the nature of the physical force acting on electrons in the ring appears to depend on one's reference frame.

This apparent paradox was how Einstein chose to introduce his 1905 paper on special relativity, titled "On the electrodynamics of moving bodies." Here's the first paragraph of his paper:

> It is known that Maxwell's electrodynamics–as usually understood at the present time–when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise–assuming equality of relative motion in the two cases discussed–to electric currents of the same path and intensity as

those produced by the electric forces in the former case.

What does Einstein mean when he says "The observable phenomenon here depends only on the relative motion of the conductor and the magnet"? To what phenomenon is he referring?

We present this discussion now to give some insight into what motivated Einstein to develop special relativity. We'll return to it later in the course and give his solution to the problem. The result is remarkable!

## 2.6 Einstein's Postulates

Einstein based his special theory of relativity on two postulates. They are as follows:

1. The principle of relativity: The laws of physics are the same in all inertial reference frames.

2. The principle of the constancy of the speed of light: The speed of light in free space has the same value $c$ in all inertial reference frames.

The first postulate says that there are no preferred inertial reference frames. One cannot ask if an object is "at rest" in an absolute sense. You may be at rest with respect to the classroom, but someone flying on an airplane with a constant velocity is equally at rest with respect to their reference frame (the airplane). Neither is "more at rest" than the other. One can only talk about the *relative* motion between observers. This postulate is consistent with Newton's laws, but not, as we have seen, with Maxwell's equations in their original form.

The second postulate seems impossible given our everyday intuition. If we think of light as a particle (photon), then we would expect the velocity of the photon to obey Galilean relativity. Imagine driving down the freeway at half the speed of light with your headlights on. An observer at the side of the road might expect the speed of the photon to be $0.5c + c = 1.5c$. But Einstein says, "no!" Both the person in the car and by the side of the road will see the photon moving at the same speed, $c$. If you think of light as a wave, then you would expect it to move at a speed $c$ relative to its medium (the hypothetical ether). But if an observer moves relative to the ether, then the speed of the wave will be greater or lower than c depending on whether she moves with or against the wave. Again, Einstein says "no!". There is no ether. Everyone see the light wave traveling at the same speed.

## 2.7 Lorentz Transformations

As mentioned above, Lorentz originally developed his transformations as a replacement to Galilean relativity based on a false assumption. However, his final result turns out to be what one finds under special relativity.

How can the speed of light be constant for all observers? The only way this is possible is if we allow space and time to "warp" into one another when two observers are in relative motion. The Lorentz transformations may be thought of as a replacement for Galilean relativity when one adopts Einstein's theory of special relativity. They describe the space and time coordinates of an object observed by two observers, one in reference frame $S$ and one in frame $S'$:

$$
\begin{aligned}
x' &= \gamma(x - ut) & x &= \gamma(x' + ut') \\
y' &= y & y &= y' \\
z' &= z & z &= z' \\
t' &= \gamma\left[t - \left(\frac{u}{c^2}\right)x\right]. & t &= \gamma\left[t' + \left(\frac{u}{c^2}\right)x'\right]. \qquad (2.4)
\end{aligned}
$$

where

$$\gamma = \frac{1}{\sqrt{1 + u^2/c^2}}$$

The left column transforms coordinates $(x, y, z, t)$ measured in frame $S$ to coordinates $(x', y', z', t')$ measured in frame $S'$. The right column represents the inverse transformation (going from frame $S'$ to frame $S$).

The symbol $\gamma$ is often called the "gamma factor". Notice that in the limit that the relative velocity between the frames is $u \ll c$, then $\gamma \to 1$. In this limit, the equation for $x'$ reduces to the Galilean transformation equation $x' = x - ut$. Also notice that in this limit, the time equation becomes $t' \approx t$. In other words, when velocities are much less than the speed of light, we "get back" the normal, everyday equations that Galileo would approve of. Space and time become "decoupled" as we normally experience them. As the relative velocity $u$ approaches the speed of light, the gamma factor blows up to infinity. Thus, the equations break down if one tried to travel at the speed of light (i.e. when $u = c$).

## 2.8 Transformation of Intervals

The Lorentz transformations may also be used to transform intervals $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$:

$$
\begin{aligned}
\Delta x' &= \gamma(\Delta x - u\Delta t) & \Delta x &= \gamma(\Delta x' + u\Delta t') \\
\Delta y' &= \Delta y & \Delta y &= \Delta y' \\
\Delta z' &= \Delta z & \Delta z &= \Delta z' \\
\Delta t' &= \gamma\left[\Delta t - \left(\frac{u}{c^2}\right)\Delta x\right]. & \Delta t &= \gamma\left[\Delta t' + \left(\frac{u}{c^2}\right)\Delta x'\right]. \qquad (2.5)
\end{aligned}
$$

## 2.9 Velocity Transformations

In Galilean relativity we derived the velocity transformation equations by directly evaluating the derivatives $dx/dt$, etc. However, because both $x$ and $t$ are a bit more complicated, evaluating this derivative would require more work. Instead, we can use the intervals to calculate an average velocity, like this:

$$v'_x = \frac{\Delta x'}{\Delta t'}$$

Substitute in for the intervals $\Delta x'$ and $\Delta t'$:

$$v'_x = \frac{\gamma(\Delta x - u\Delta t)}{\gamma\left[\Delta t - \left(\frac{u}{c^2}\right)\Delta x\right]}$$

Cancel the gamma factors:

$$v'_x = \frac{\Delta x - u\Delta t}{\Delta t - \left(\frac{u}{c^2}\right)\Delta x}$$

Divide the top and bottom of this equation by $\Delta t$ and substitute $v_x = \Delta x/\Delta t$ to give

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}.$$

Repeating for $v'_y$ and $v'_z$ gives the the following **Lorentz velocity transformation equations**:

$$\begin{aligned} v'_x &= \frac{v_x - u}{1 - v_x u/c^2} & \qquad v_x &= \frac{v'_x + u}{1 + v'_x u/c^2} \\ v'_y &= \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} & \qquad v_y &= \frac{v'_y\sqrt{1 - u^2/c^2}}{1 + v'_x u/c^2} \\ v'_z &= \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2} & \qquad v_z &= \frac{v'_z\sqrt{1 - u^2/c^2}}{1 + v'_x u/c^2}. \end{aligned} \tag{2.6}$$

The left column solves for the velocities in the $S'$ frame while the right column solves the for $S$ velocities. These may be thought of as inverse transformations of one another. Notice that to go from the left column to the right column one swaps primed with unprimed variables and replaces the $u$ with $-u$. Why does this work?

In the limit that $u \ll c$, we once again recover the Galilean transformation equations, i.e. $v'_x \approx v_x - u$, $v_y \approx v'_y$, $v'_z \approx v_z$.

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**Example 3. Relative Velocities (Using Relativity).**

Let's recalculate Example 1, this time using relativity. As in most relativity problems, it is best to express velocities in terms of the speed of light. We calculate the following:

- $u = 20$ mph $= 2.982 \times 10^{-8}c =$ the velocity of Mileva relative to Albert
- $v_x = 50$ mph $= 7.456 \times 10^{-8}c =$ the velocity of the Porsche relative to Albert
- $v'_x =$ the velocity of the Porsche relative to Mileva

Using the Lorentz velocity transformation gives

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}.$$

On my calculator, the denominator is so close to unity that I just get $v'_x \approx v_x - u = 30$ mph, which is the same result we found using Galilean relativity. If we wanted a super accurate result, we could use the binomial approximation $(1 - x)^{-1} \approx 1 + x$ to write

$$\frac{1}{1 - v_x u/c^2} \approx 1 + v_x u/c^2.$$

We find that this factor is

$$\frac{1}{1 - v_x u/c^2} \approx 1 + (7.456 \times 10^{-8}c)(2.982 \times 10^{-8}c)/c^2 = 1 + 2.22 \times 10^{-15}$$

or

$$\frac{1}{1 - v_x u/c^2} \approx 1.00000000000000222.$$

Multiplying this factor by the numerator gives

$$v'_x \approx 30 \text{ mph} + 6.67 \times 10^{-14}\text{mph}.$$

We see that relativity isn't important for normal, everyday speeds, unless you are interested in very, very, very accurate results.

**Example 4. Another Relative Velocity Problem.**

Let's try an example using velocities closer to the speed of light. Let's assume Mileva passes Albert at half the speed of light ($0.5c$) and Mileva sees the Porsche pass her at 90% the speed of light ($0.59c$). How fast is the Porsche moving relative to Albert?

**Solution.** This time we want to find $v_x$:

- $u = 0.5c =$ the velocity of Mileva relative to Albert
- $v'_x = 0.9c =$ the velocity of the Porsche relative to Mileva
- $v_x =$ the velocity of the Porsche relative to Albert

We plug these values directly into the inverse velocity transformation equation for $v_x$:

$$v_x = \frac{v'_x + u}{1 + v'_x u/c^2}$$

so,

$$v_x = \frac{0.9c + 0.5c}{1 + (0.9c)(0.5c)/c^2} = 0.966c.$$

While Galilean relativity would predict that Albert sees the Porsche moving at $1.4c$, the Lorentz transformation shows that Albert "only" see it moving at $0.966c$. The velocity addition formula says that relative velocities will *always* be less than the speed of light! If one observer sees an object moving at less than the speed of light, all observers will see it moving at less than the speed of light. As we will see in a later lecture, the speed of light represents an ultimate speed limit in the universe.

**Example 5. Two Observers Watch a Beam of Light.**

What happens to the velocity transformation equations when we imagine looking at a beam of light? Let's imagine Albert is driving with his headlights on. He sees the light leaving his headlights at speed $c$. How fast will Mileva see the light beam?

**Solution.** We set $v_x = c$ and we want to solve for $v'_x$. This time, let's just call Mileva's speed relative to Albert $u$, i.e. we'll leave it general. The Lorentz velocity transformation equation gives

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}$$

We plug in $v_x = c$ to find

$$v'_x = \frac{c - u}{1 - cu/c^2}.$$

Multiplying top and bottom by $c$ gives

$$v'_x = \frac{c - u}{c - u}c.$$

Simplifying gives

$$v'_x = c.$$

Thus, Mileva sees the light beam traveling at the speed of light, just like Albert. This is of course the result we should have gotten since Einstein's second postulate was that all observers will measure the velocity of light to be $c$, independent of the velocity of the object that produced the light. This result is true for any velocity $|u| < c$. What happens when $u = c$?

**Example 6. Rain Problem Revisited.**

Let's rework the rain problem in Example 2, this time using relativity and assuming the rain drops are photons falling at the speed of light. We want to find the velocity components of the photons in Mileva's reference frame, assuming she is moving along the $x$ axis relative to Albert.

**Solution.** Since Albert sees the photon falling straight down, we have $v_x = 0$, $v_y = -c$ and $v_z = 0$. We want to find the photon velocity in Mileva's frame, i.e. $\vec{v}' = (v'_x, v'_y, v'_z)$. The velocity transformation equation for $v'_x$ gives:

$$v'_x = \frac{v_x - u}{1 - v_x u/c^2}.$$

Plug in $v_x = 0$:

$$v'_x = -u.$$

The $v'_y$ component is

$$v'_y = \frac{v_y\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$$

Plug in $v_y = -c$ and $v_x = 0$:

$$v'_y = -c\sqrt{1 - u^2/c^2}$$

The $v'_z$ component is

$$v'_z = \frac{v_z\sqrt{1 - u^2/c^2}}{1 - v_x u/c^2}$$

Plugging in $v_z = 0$ gives

$$v_z = 0.$$

Thus the velocity vector of the photon is

$$\vec{v}' = (-u, -c\sqrt{1 - u^2/c^2}, 0).$$

Notice that the speed of the photon is

$$v' = \sqrt{v_x'^2 + v_y'^2 + v_z'^2}$$

$$v' = \sqrt{u^2 + c^2(1 - u^2/c^2) + 0}$$

or

$$v' = c.$$

Thus we find the Mileva sees the photon moving at the speed of light as she should.

The aberration of the light is defined as the deviation of the angle from what Albert sees (in this case, the angle measured from straight up, along the $+y$ axis). We can calculate it as

$$\tan\alpha = \frac{-v_x'}{-v_y'} = \frac{u}{c\sqrt{1 - u^2/c^2}}.$$

We can simplify this expression by substituting $\gamma = 1/\sqrt{1 - u^2/c^2}$ to find

$$\tan\alpha = \gamma\frac{u}{c}.$$

If Mileva's velocity relative to Albert is much less than the speed of light, then $\tan\alpha \approx \alpha$ and $\gamma \approx 1$ so

$$\alpha \approx \frac{u}{c},$$

consistent with the result in Example 2 for Galilean relativity. In the opposite limit, when Mileva's velocity relative to Albert approaches the speed of light, we find $\gamma \to \infty$ so $\alpha \to 90°$. In this case, the light looks like it is coming from directly in front of her instead of above her.

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