Homework 5

Goal: To gain experience numerically integrating ODEs using Euler's method.

Due: April 4

1. (10 points) (a) Use a calculator and Euler's method to numerically integrate the following differential equation by hand:
$$\frac{dx}{dt} = ax.$$
Assume $a = -0.1$ and $x_0 = 1$. Integrate the equation from $t = 0$ to $t = 10$ with $\Delta t = 2$. Create a table showing the iteration number n, time t_n, and position x_n.

   (b) Does this combination of a and Δt give a stable integration scheme? Base your answer on the stability analysis technique described in class.

2. Write a Matlab program to numerically integrate the following differential equation on the interval $0 \le t \le t_{\max}$ using Euler's method:
$$\frac{dx}{dt} = ax,$$
where a is a constant. The program should prompt the user to enter the following parameters:

   - maximum time $t_{\max}$
   - step size Δt
   - coefficient a
   - initial condition x_0

   After numerically integrating the differential equation, your program should output two subplots. In the first subplot, graph the numerical result $x(t)$ and the exact solution
$$x_{\text{exact}} = x_0 e^{at}.$$
Use a legend to label each. In the second subplot, graph the error between the numerical and the exact solutions. In other words, plot $x - x_{\text{exact}}$ v.s. time.

   Run your program for the following cases and save your plots as P02a.pdf, P02b.pdf and P02c.pdf:

   (a) $a = -5$, $t_{\max} = 10$, $\Delta t = 0.5$.

   (b) $a = -5$, $t_{\max} = 10$, $\Delta t = 0.1$.

   (c) $a = -5$, $t_{\max} = 10$, $\Delta t = 0.01$.

   Which examples are numerically stable? Which are unstable? Calculate the maximum time step that gives a numerically stable integration scheme. Estimate the maximum numerical error from each graph. How does reducing the step size Δt affect the numerical error?

3. In this problem you will model the dynamics of an oscillating mass m on a spring with spring constant k. The differential equation describing the motion is
$$\frac{d^2x}{dt^2} = -\omega_0^2 x,$$
where $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of oscillations. Write a Matlab program to numerically integrate the equation using Euler's method over at least 4 periods. The program should produce three plots:

(a) Position v.s. time graph of the numerical solution and the exact solution.

(b) Error ($x - x_{\text{exact}}$) v.s. time. The exact solution when the mass is released from rest is

$$x_{\text{exact}}(t) = x_0 \cos(\omega_0 t)$$

(c) The total energy of the system as a function of time given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Pick the initial conditions x_0 and v_0. Choose two time steps (one at least 10 times smaller than the other). How does changing the time step affect the error and energy conservation? Does the numerical solution conserve energy?