The Schrödinger Equation (Part 2)
Finite Potential Well and Step Barriers

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial\Psi}{\partial t}$$

Kinetic Energy + Potential Energy = Total Energy

Method for Solving Time-Independent Schrödinger Equation (SE)

Step 1. Define potential U(x), draw a picture and plug into SE

Step 2. Solve SE or guess general solutions to the SE.

Step 3. Write down boundary conditions.

Step 4. Use boundary conditions to place constraints on unknown constants and coefficients.

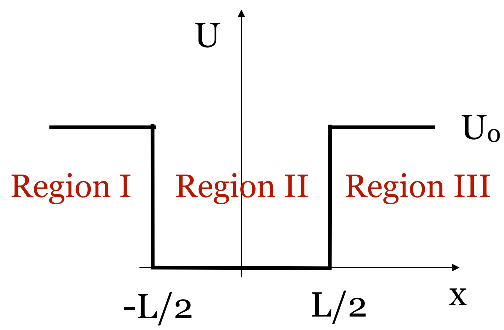
Step 5. Use boundary constraints to solve for energy levels E_n

Step 6. Use normalization to solve for remaining unknown constants and write down final wave functions $\psi_n(x)$ associated with each energy level E_n

Finite Potential Well

step 1 Define the potential

$$U(x) = \begin{cases} U_0 & \text{for } x \leq -L/2 \quad \text{(Region I)} \\ 0 & \text{for } -L/2 < x < L/2 \quad \text{(Region II)} \\ U_0 & \text{for } x \geq L/2 \quad \text{(Region III)} \end{cases}$$



Solve general case of constant potential U_o

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}(E - U_0)\psi(x)$$

General Constant Potential: $U(x) = U_o$

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}(E - U_0)\psi(x)$$

Case I: $E > U_0$ $\quad \psi(x) = A\sin(kx) + B\cos(kx)$ or

$$\psi(x) = Ce^{ikx} + De^{-ikx}$$

where: $k = \dfrac{\sqrt{2m(E - U_0)}}{\hbar}$

Classically Allowed

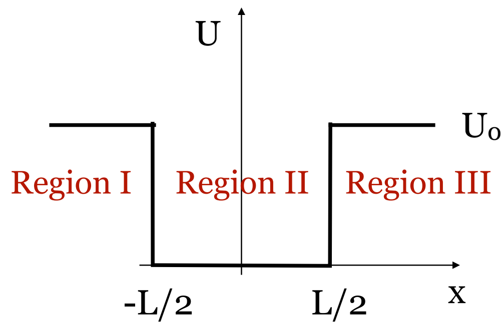
Case II: $E < U_0$ $\quad \psi(x) = A'e^{k'x} + B'e^{-k'x}$

where: $k' = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$

Classically Forbidden

Finite Potential Well

step 2



Region I $\quad \psi_I(x) = Ce^{k'x} + De^{-k'x}$

Region II $\quad \psi_{II}(x) = A\sin(kx) + B\cos(kx)$

Region III $\quad \psi_{III}(x) = Ee^{k'x} + Fe^{-k'x}$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

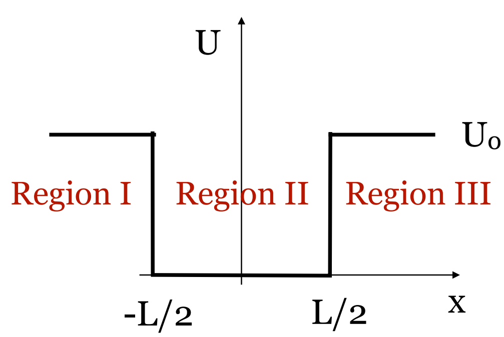
$$k' = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

Finite Potential Well

step 3 Boundary conditions

$\psi_I(-\infty) = 0$

$\psi_{III}(\infty) = 0$



$$\psi_I(-L/2) = \psi_{II}(-L/2)$$

$$\left.\frac{d\psi_I}{dx}\right|_{x=-L/2} = \left.\frac{d\psi_{II}}{dx}\right|_{x=-L/2}$$

$$\psi_{II}(L/2) = \psi_{III}(L/2)$$

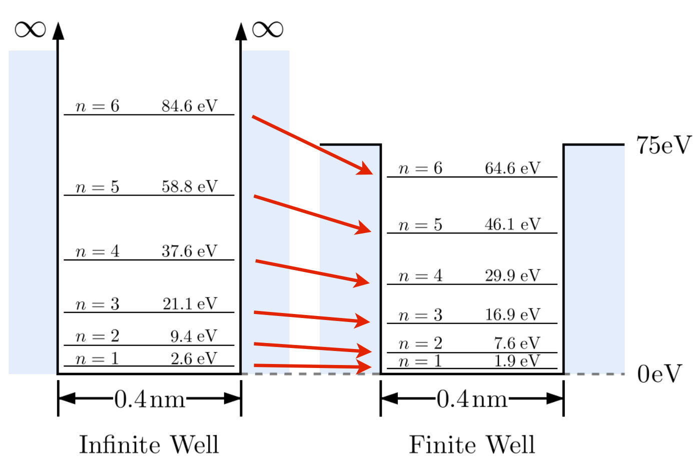
$$\left.\frac{d\psi_{II}}{dx}\right|_{x=L/2} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L/2}$$

Finite Potential Well

step 4 Use B.C. to constrain unknown constants

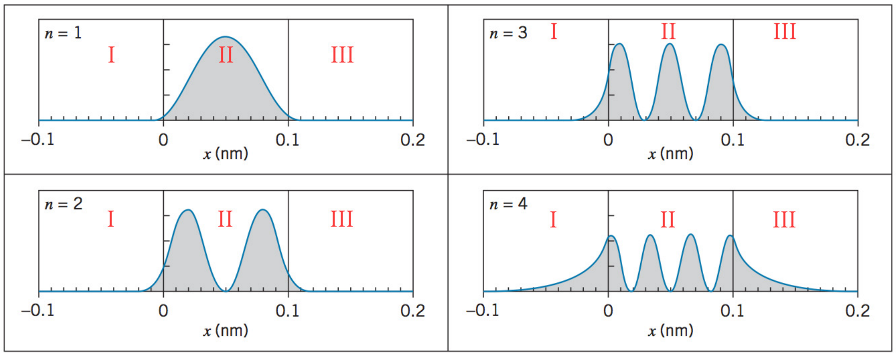
Finite Potential Well

step 5 Need to use numerical techniques to solve for energies. Here is the result for a specific well size.



Finite Potential Well

step 6 Again, we just present the graphs.



Note that the wave function is oscillatory in the classically allowed region and exponentially decays in the classically forbidden regions. Also note that the wavelengths "spread out" slightly compared with the infinite potential well.

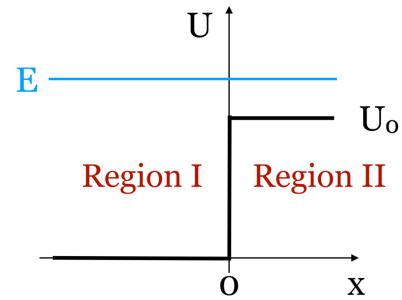
General Behavior of Wave Functions in a Potential Well

1) $\psi(x)$ is oscillatory in classically allowed regions where $E > U$
2) $\psi(x)$ decays in classically forbidden regions where $E < U$
3) Wavelength decreases as K.E. increases
4) Amplitude increases as probability of finding particle increases

Step Barrier: E > U_o

step 1 Define the potential

$$U(x) = \begin{cases} 0 & x < 0 \\ U_o & x \geq 0 \end{cases}$$



$$\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}E\psi(x)$$ Region I

$$\frac{\partial^2\psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}(E - U_0)\psi(x)$$ Region II

General Constant Potential: U(x) = U_o

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Classically Allowed

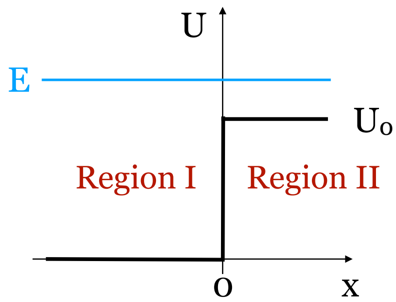
Case II: $E < U_0$ $\quad \psi(x) = A'e^{k'x} + B'e^{-k'x}$

where: $k' = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$

Classically Forbidden

General Constant Potential: U(x) = U₀

step 2

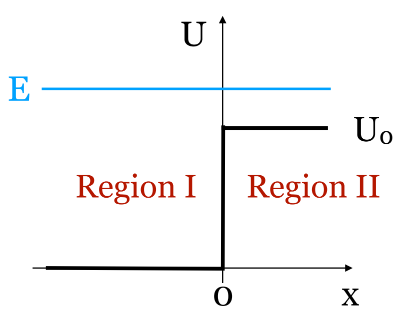


Region I $\quad \psi_I(x) = A'e^{ik_0x} + B'e^{-ik_0x} \qquad k_0 = \dfrac{\sqrt{2mE}}{\hbar}$

Region II $\quad \psi_{II}(x) = C'e^{ik_1x} + D'e^{-ik_1x} \qquad k_1 = \dfrac{\sqrt{2m(E-U_0)}}{\hbar}$

General Constant Potential: U(x) = U₀

step 2



Region I $\quad \psi_I(x) = Ae^{ikx} + Be^{-ikx} \qquad k = \dfrac{\sqrt{2mE}}{\hbar}$

Region II $\quad \psi_{II}(x) = Ce^{ik'x} + De^{-ik'x} \qquad k' = \dfrac{\sqrt{2m(E-U_0)}}{\hbar}$

$$\Rightarrow \quad \Psi_I(x,t) = \psi_I(x)e^{-i\omega t} = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

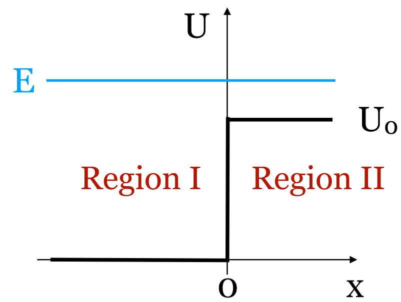
$$\Psi_{II}(x,t) = \psi_{II}(x)e^{-i\omega t} = Ce^{i(k'x-\omega t)} + De^{-i(k'x+\omega t)}$$

Traveling waves

General Constant Potential: $U(x) = U_o$

step 3 Boundary Conditions

(HW problem 28)



General Constant Potential: $U(x) = U_o$

step 5 Energies are NOT quantized (why?)

step 6 Wave functions are NOT normalized (why?)

Instead, solve for $\dfrac{|B'|^2}{|A'|^2}$ and $\dfrac{|C'|^2}{|A'|^2}$