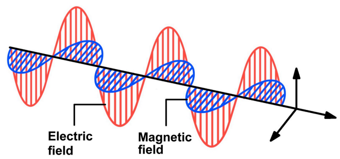
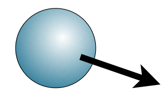
Chapter 4: Wave Properties of Particles

Light

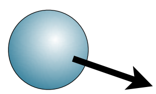


EM Wave

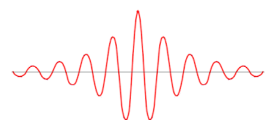


Photon

Matter



Electron



wave function

de Broglie Wave - 1924

Relativistic equation for massless particle: $E = \sqrt{(pc)^2 + (\cancelto{0}{mc^2})^2}$

$$E = pc$$

Energy of a photon: $E = hf = \dfrac{hc}{\lambda}$

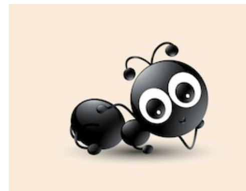
Combine: $\dfrac{hc}{\lambda} = pc \;\rightarrow\; \boxed{\lambda = \dfrac{h}{p}}$ de Broglie wavelength

de Broglie suggests this wavelength applies to both massless photons and particles with mass (Nobel Prize 1929)

Example: de Broglie wavelength of an ant

mass = 0.1 g $= 10^{-4}$ kg

velocity = 1 cm/s $= 10^{-2}$ m/s

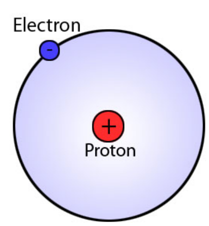


Find the ant's de Broglie wavelength

Example: de Broglie wavelength of an electron in atom

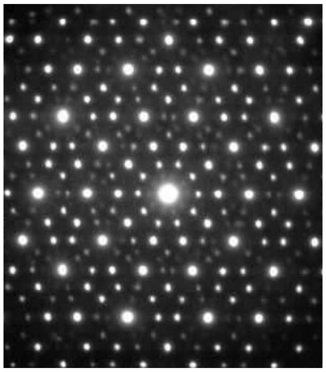
KE ≈ 1 eV

m = 0.511 MeV

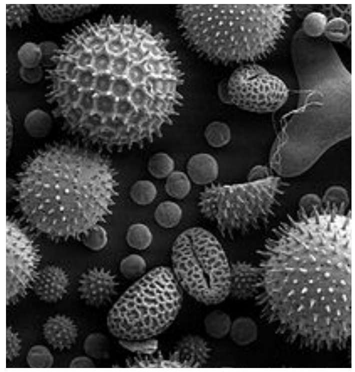


Find the electron's de Broglie wavelength

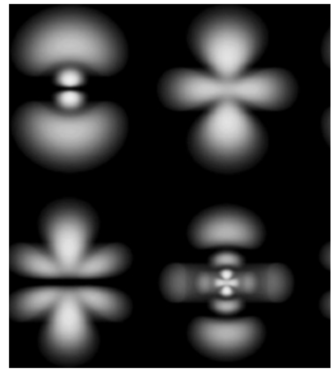
Evidence for de Broglie Waves



diffraction of electrons through a crystal



electron microscope resolution



quantum wave mechanics

“h bar”

We define the following: $\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}$ J s

Why? It comes up a lot. For example:

Write momentum in terms of wavenumber $k = \frac{2\pi}{\lambda}$

$$\boxed{p = \hbar k}$$

Frequency of a de Broglie wave

For waves: $\lambda f = v_p$ where v_p is the phase velocity

Problem: we don't know what v_p is

Instead, we try the expression for energy of a photon: $\boxed{E = hf}$

Write in terms of the angular frequency: $\omega = 2\pi f$

$$\Rightarrow \quad \boxed{E = \hbar\omega}$$

Energy and Momentum of a de Broglie wave

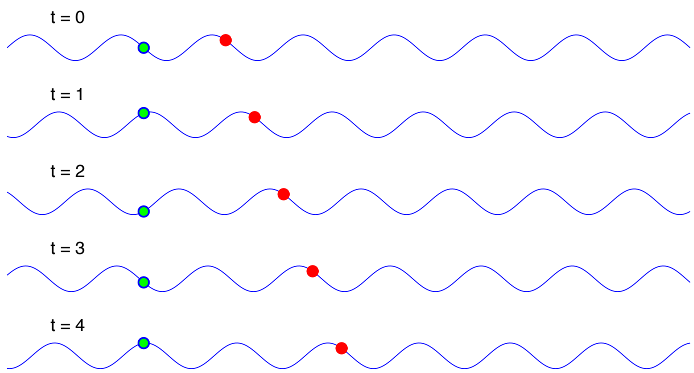
$$p = \frac{h}{\lambda} \qquad p = \hbar k$$

$$E = hf \qquad E = \hbar\omega$$

All physical properties of a particle are "encoded" in the wave function.

Traveling Wave

$$\psi(x,t) = A\sin(kx - \omega t) \qquad k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f$$



Phase Velocity

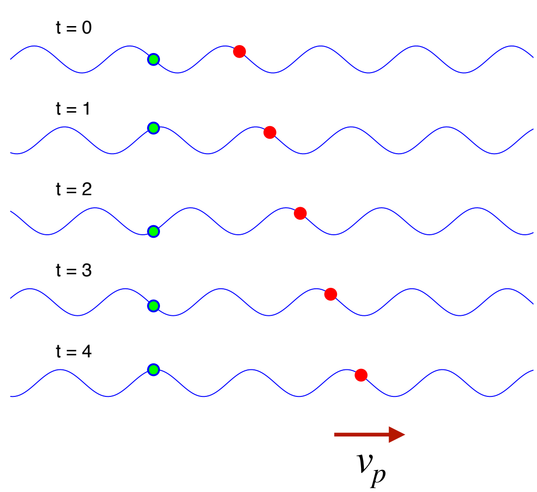
Pick a point on the wave and track it so phase remains constant

$$kx - \omega t = \phi_0 = \text{const.}$$

$$x = \frac{\phi_0}{k} + \frac{\omega}{k}t$$

$$x = x_0 + v_p t$$

$$\boxed{v_p = \frac{\omega}{k} = \lambda f}$$



Problem: Find the phase velocity of a de Broglie wave in terms of the particle velocity v

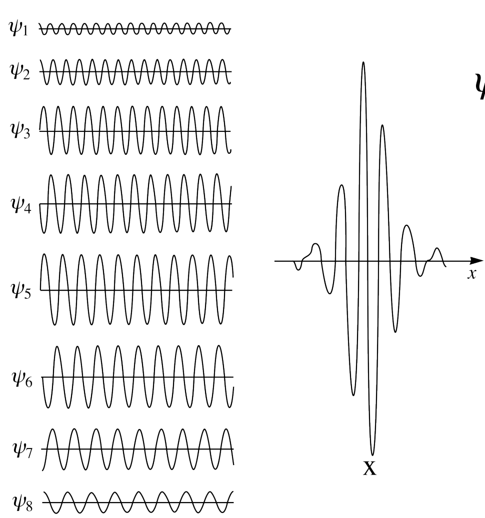
use the definition: $v_p = \dfrac{\omega}{k} = \lambda f$

Answer: $v_g = \left(\dfrac{c}{v}\right) c$

always greater than the speed of light!!

Wave Packet

Wave packets are constructed from the superposition of two or more traveling waves



$$\psi(x,t) = \sum_i A_i \sin(k_i x - \omega_i t)$$

Superposition of two traveling waves

$$\psi(x,t) = A_1 \sin(k_1 x - \omega_1 t) + A_2 \sin(k_2 x - \omega_2 t)$$

First consider two purely spatial waves with equal amplitudes:

$$\psi(x,t) = A \sin(k_1 x) + A \sin(k_2 x)$$

Use trig identities to show

$$\psi(x,t) = 2A \cos\left[\frac{1}{2}(k_1 - k_2)x\right] \sin\left[\frac{1}{2}(k_1 + k_2)x\right]$$

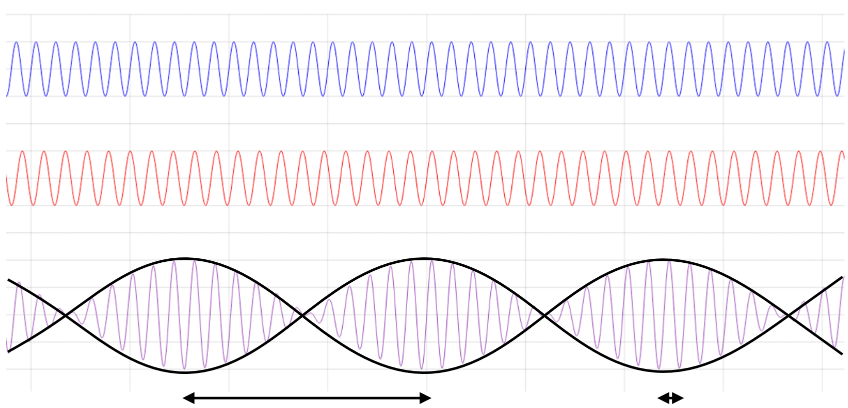
Low frequency modulating envelope

$k_{\text{beat}} = (k_1 - k_2)$

High frequency carrier wave

$k_{\text{carrier}} = \frac{1}{2}(k_1 - k_2)$

Beats



$k_{\text{beat}} = (k_1 - k_2)$ $\qquad$ $k_{\text{carrier}} = \frac{1}{2}(k_1 - k_2)$

Superposition of two traveling waves

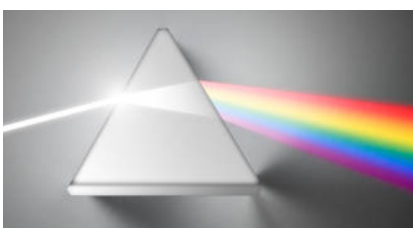
$$\psi(x,t) = A_1 \sin(k_1 x - \omega_1 t) + A_2 \sin(k_2 x - \omega_2 t)$$

Phase velocity (of carrier wave) $\quad v_p = \dfrac{\omega_1 + \omega_2}{k_1 + k_2}$

Group velocity (of modulating wave) $\quad v_g = \dfrac{\omega_1 - \omega_2}{k_1 - k_2}$

Dispersion Relation

In a dispersive medium, the phase velocity varies as function of the wavenumber. The function $\omega(k)$ is called the dispersion relation.



Group Velocity

Velocity of the envelope of a wave packet.

$$\boxed{v_g = \frac{d\omega}{dk}}$$

Problem: Find the group velocity of a de Broglie Wave

Use the definition: $v_g = \dfrac{d\omega}{dk}$

Answer: $v_g = v$

Problem: An electron has a de Broglie wavelength of 2 pm. Find: k, p, v_p, v_g

Momentum: $p = \frac{h}{\lambda} = \frac{hc}{\lambda}\frac{1}{c} = \frac{1240 \text{ eV nm}}{2 \times 10^{-3} \text{ nm}}\frac{1}{c} = 0.62 \text{ MeV / c}$

Kinetic Energy:

$$E = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{(0.62 \text{ MeV / c})^2 + (0.511 \text{ MeV})^2} = 0.803\text{MeV}$$

$$K = E - mc^2 = 0.803\text{MeV} - 0.511 \text{ MeV} = 0.292 \text{ MeV}$$

Phase velocity: $v_p = \frac{c}{v}c$

need v: $E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$ $\qquad \frac{v}{c} = \sqrt{1 - \left(\frac{mc^2}{E}\right)} = 0.771$

$$v_p = \frac{1}{0.771}c = 1.3\ c$$

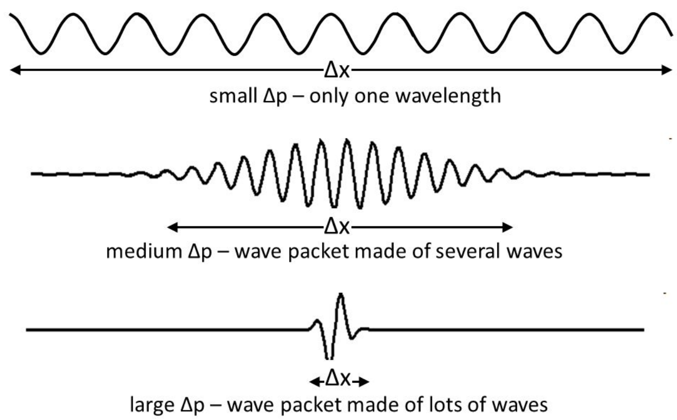
Problem: An electron has a de Broglie wavelength of 2 pm. Find: k, p, v_p, v_g

Group Velocity: $v_g = v = 0.771\,c$

Spatially Localized Wave Packet

Add together an infinite number of waves with amplitudes *g(k)*

$$\psi(x,t) = \int_0^\infty g(k)\cos(kx - \omega t)dk$$



Spatially Localized Wave Packet

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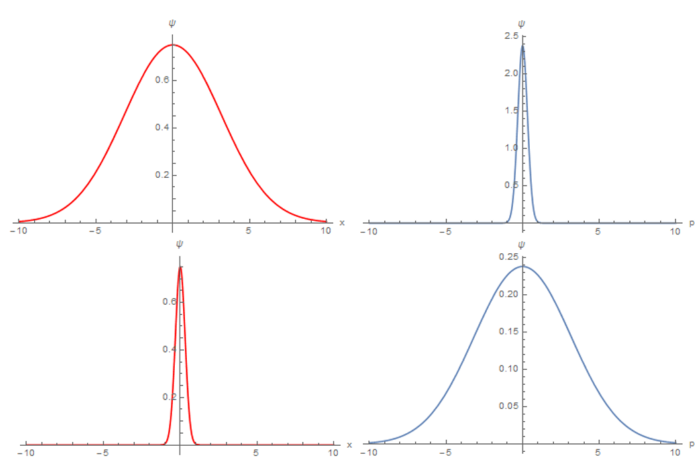

Heisenberg Uncertainty Principle

It is not possible to make a simultaneous determination of the position and the momentum of a particle with unlimited precision.

$$\Delta x \Delta p_x \geq \frac{1}{2}\hbar$$

It is not possible to make a simultaneous determination of the enerty and the time coordinate of a particle with unlimited precision.

$$\Delta E \Delta t \geq \frac{1}{2}\hbar$$

Particles can act like waves.
But what waving?

Classical waves: Power $\propto$ Amplitude^2

Quantum waves: Probability $\propto$ Amplitude^2

Prob $\propto |\psi|^2$