Lecture 3. Implications of Special Relativity

Contents

Summary. In the previous lecture we saw that under special relativity the Galilean transformations were replaced by the Lorentz transformations. In this lecture, we look at some of the implications of special relativity.

2.1 Spacetime Diagrams

A **spacetime** diagram is like a normal position-time plot with two exceptions. First, the axes are swapped so that time is on the vertical axis and space is on the horizontal axis. In a position-time plot, the object's velocity is the slope of the graph. In a spacetime diagram the velocity is one over the slope (1/slope). Second, the time axis is scaled by the speed of light so that ct is plotted rather than t. This scaling is convenient because light rays will then move along diagonal lines at a 45° angle (why?). In a spacetime diagram, the path of an object is often called the object's **worldline**. Figure 1 shows some examples. A particle at rest moves vertically upward (the worldline has infinite slope and thus zero velocity). When the slope is greater than one, the particle has a speed less that c and when its slope is greater than one, it has a velocity greater than c (which is impossible in special relativity).

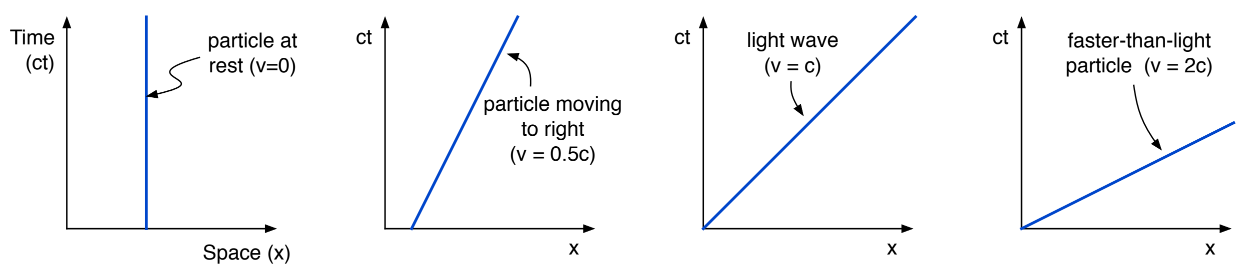


Figure 1: Spacetime diagrams showing the worldlines of particles moving at different speeds.

A particle-like **object** will always trace out a world line in a spacetime diagram. An **event** is associated with some abrupt occurrence in spacetime: you snap your fingers, your alarm goes off, you say "hi" to your friend. An event by itself has no velocity so it doesn't make sense to ask if it is “at rest” with respect to a given reference frame. A single event is is only defined for an instant in time.

2.2 Relativity of Simultaneity

Two events are said to be **simultaneous** if they occur at the same time. For instance if you and your friend happen to say the same word (for example, “physics”) at exactly the same time, then we take it for granted that *everyone* would agree that these two events happened simultaneously. However, special relativity says that simultaneity is relative, i.e. two observers must be at rest in the same inertial reference frame for them to agree that two events happen simultaneously. If they aren’t at rest with respect to each other, one observer will see one event happening either slightly before or after the other.

We introduce the relativity of simultaneity through a **thought experiment**. Thought experiments are widely used in relativity to understand the logical consequences of the nonintuitive postulates of special relativity. Einstein was well-known for them.

Let us imagine that a Republican Senator and a Democratic Senator have agreed to a bipartisan bill. However, neither wants to sign the bill first to avoid looking weak. On the day of the signing, they sit on opposite ends of a train car, the Republican sits in the front, the Democrat sits in the back. They each agree to sign their own copy of the bill once a light flashes in the middle of the train car. As part of ceremony, the car will pass a train station where reporters wait on the platform to document the event.

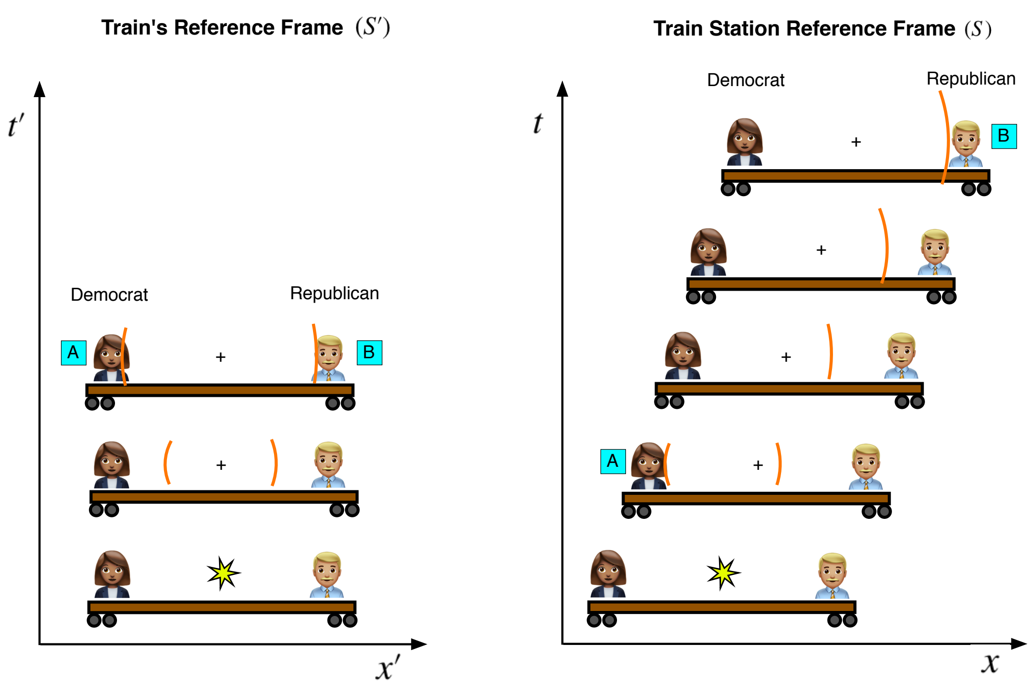


Figure 2: Illustration demonstrating relativity of simultaneity.

Let’s first see what happens from the point of view of the senators riding on the train. Figure 2 shows the sequence of events. At the instant the light flashes, light waves will travel outward, toward both the Democrat and the Republican. Because they are equally distant from the light and because all observers see light travel at the same speed, c, each senator will each see the light turn on at the same time. We can define two events. Event A is when the light wave reaches the Democrat at the back of the train. Event B is when the light wave reaches the Republican at the front of the train. These events happen simultaneously in the train's frame of reference. From the perspective of the senators on the train, they sign the bill at exactly the same time and they congratulate each other on bringing bipartisanship to Washington.

The reporters standing at the train station see things quite differently, however. They clearly see that event A (the signing of the bill by the Democratic senator) happens *before* event B (the signing of the bill by the Republican senator). They report that the Republican tricked the Democrat into signing first. Why did the reporters see event A before event B? In this frame, the Democrat is moving forward, toward the light wave that is coming toward her. She will meet the light wave somewhere "in the middle". Since the light wave doesn't have to travel as far, the reporters will see it arrive at the Democratic senator sooner than observers in the train's reference frame. Because the Republican is moving away from the light wave, the wave must travel farther to catch up to him. Thus, the reporters will see the wave catch up to him later than observers in the train frame. As shown in Figure 2, the events A and B do not occur simultaneously in the train station's frame.

Figure 3 shows the same scenario using a spacetime diagram and worldlines.

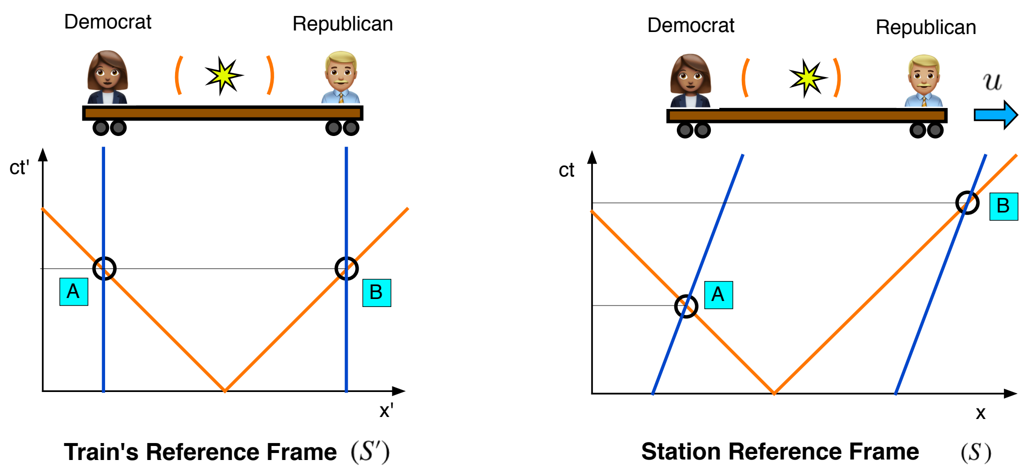


Figure 3: Illustrations showing a thought experiment of two astronomers riding on a train.

One might ask: which perspective is *correct*? Do the senators sign the bill at the same time or don't they? The answer is that both perspectives are equally valid. *Simultaneity is relative.* Things that remain the same in all reference frames are said to be **invariant.** Thus, simultaneity is *not* invariant.

Problem 1. Newtonian Simultaneity. Why don't we notice these effects in everyday life? Analyze this example from a Newtonian perspective using Galilean relativity. Do the senators and the reporters both agree that events A and B happen simultaneously? If so, explain why. Draw a spacetime diagram of the events in the train station reference frame. If it helps, you can think of the light as two particles that move outward from the light source.

Problem 2. Show that the slope of the line connecting events A and B turns out to be c/u, where u is the velocity of the train. Hint: let $x = 0$ be at the light right when it flashes. Write down equations for each world line and solve for the coordinates (x, ct) of each event.

2.3 Minkowski Diagram

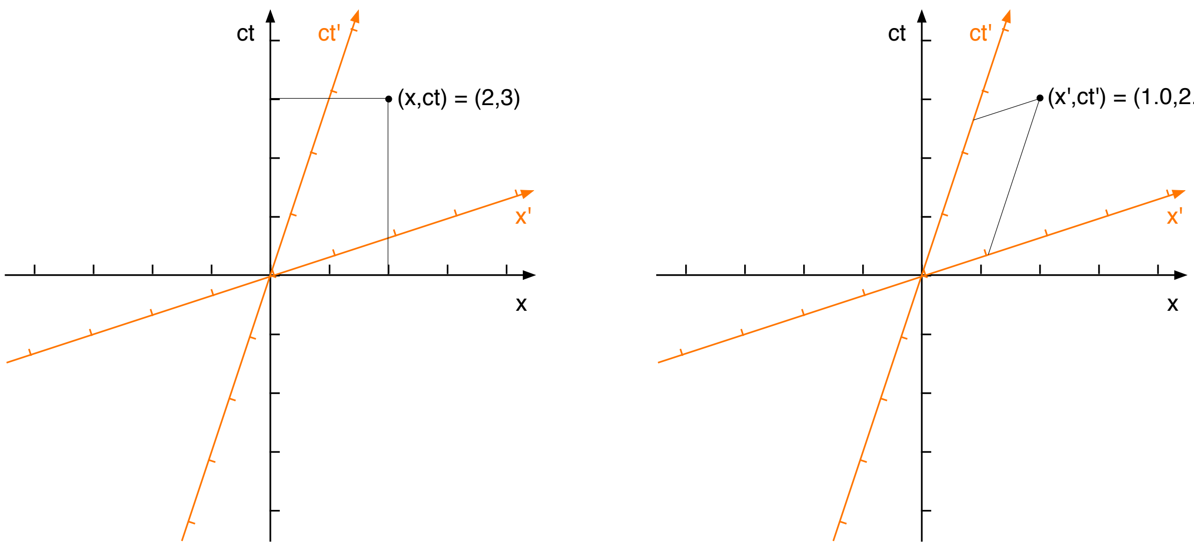


Figure 4: Reading the coordinates of an event from the S (left) and S' (right) frames.

A **Minkowski diagram** is a spacetime diagram used to demonstrate the effects of special relativity by superimposing coordinate systems for both the S and S' frames on the same diagram. Figure 4 shows an example. The black (x, ct) axes show a "normal" spacetime coordinate system. The orange (x', ct') axes showing the S' frame are warped according to the Lorentz equations. As usual, we assume the S' frame is moving with relative velocity u with respect to the S frame. There are a few unique things to notice about this warped coordinate system:

- The ct' axis has a slope c/u and the x' axis has slope u/c, i.e. their slopes are the inverse of each other. When the relative velocity $u = 0$, both pairs of axes fall on top of each other. As u increases, the ct' axis tilts farther away from the vertical, and the x' axis tilts upwards away from the horizontal. In the limit that $u \to \infty$, both axes approach the 45° diagonal.

- The unit tick marks on the ct' and x' axes increase in length as u increases. As u changes, the unit tick marks on traces out a hyperbola according to $1 = (\Delta x)^2 - (c\Delta t)^2$ for the x' unit vector and $1 = (c\Delta t)^2 - (\Delta x)^2$ for the xct' unit vector.

- To read off the ct' coordinates of an event, one draws a line parallel to the x' axis that passes through the event. The intersection of this line with the ct' gives the desired value. Similarly, to read off the x' coordinate, one draws a line through the event parallel to the ct' axis. (See Figure 4).

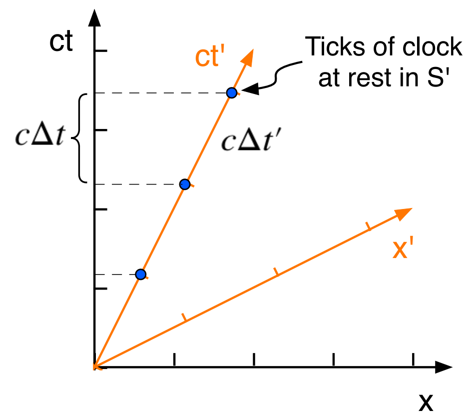
2.4 Time Dilation

When two observers are in relative motion, each will see time run more slowly for the other. This effect is called **time dilation**.

Imagine that two famous astronomers, Vera Ruben and Jocelyn Bell, are doing experiments to test special relativity. Vera holds a clock that ticks once a second. Let's call this reference frame S'. By definition the interval between ticks is $\Delta t' = 1$ s. Because the clock is at rest with respect to her, its displacement between ticks is $\Delta x' = 0$.

Now, let's assume Jocelyn rides a bicycle past the clock. Her velocity relative to the clock is u. Let's call Jocelyn's frame S. She records the positions $x_1, x_2, x_3, ...$ and times $t_1, t_2, t_3, ...$ of the clock when it ticks. She calculates the time interval between ticks as $\Delta t = t_{i+1} - t_i$ and the displacement between ticks as $\Delta x = x_{i+1} - x_i$.

To recap, we know $\Delta x' = 0$ and we want to find Δt in terms of $\Delta t'$. We use the Lorentz transformation of intervals for Δt:



$$\Delta t = \gamma \left[\Delta t' + \left(\frac{u}{c^2} \right) \Delta x' \right].$$

Since $\Delta x' = 0$, this immediately becomes

$$\Delta t = \gamma \Delta t' = \frac{1}{\sqrt{1 - u^2/c^2}} \Delta t'.$$

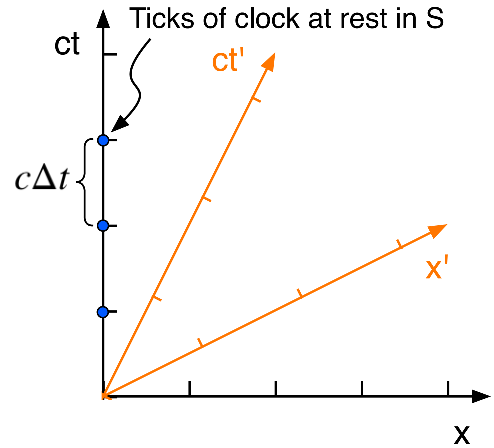
Thus, we see that Jocelyn sees Vera's clock running slow by a factor γ. When $u \ll c$, the gamma factor γ is close to unity and $\Delta t \approx \Delta t'$. However, as u starts to become a significant fraction of the speed of light, the time dilation factors can be large. In the limit that $u \to c$, we see that $\gamma \to \infty$ and the time between ticks on the moving clock becomes infinite, i.e. time essentially "stops."

The **proper time** interval τ_0 of a pair of events is the time between the events measured by an observer who is present at each event. One could also say that for an observer to measure the proper time between two events, the events must be at the same location in her reference frame. In the above example, the proper time interval is $\tau_0 = \Delta t'$. The nonproper time interval τ is the time between the events for an observer in relative motion to the proper time frame. In the above example, $\tau = \Delta t$. Thus, we could write

$$\tau = \gamma \tau_0. \tag{2.1}$$

This equation is simpler than the Lorentz equation because it assumes that the displacement between the events in one frame (the proper frame) is zero. The proper time is the shortest possible time between two events. All other reference frames will measure a longer time interval between them.

Problem 3. Consider reversing the S and S' frames in the above time dilation example. Let Vera and the clock's frame be S and Jocelyn's frame be S'. The figure at right shows the ticking clock in the S frame. (a) draw lines on this diagram to graphically show time dilation in the S' frame. (b) Derive an expression for $\Delta t'$ in terms of Δt.



2.5 Length Contraction

When an observer is moving relative to an object, the observer will measure the object to be contracted along its direction of motion. This effect is called **length contraction**.

Now let's assume Vera (in S' frame) and Jocelyn (in S frame) are flying identical spaceships, each traveling along the x axis. Both Vera and Jocelyn measure their own spaceships to have length L_0. The length of an object measured in its own reference frame is often called the **proper length**. We'll assume Jocelyn flies past Vera with relative velocity u. How long is Vera's ship as observed by Jocelyn?

In order to apply the Lorentz interval transformations, we need to define two events to define a space and time interval. Let's imagine that Jocelyn takes a photo of Vera's ship as it flies by. We measure the length of the ship by measuring the distance between the back of the ship and the front of the ship at the instant the photo is taken (in Jocelyn's S frame). The first event is the act of photographing the back of the ship and the second event is the act of photographing the front of the ship. In symbols, we can say $\Delta t = 0$ (since the photo captures the front and back simultaneously) and $\Delta x' = L_0$. We want to find Δx.

We want a Lorentz transformation equation that contains Δt, Δx and $\Delta x'$. The winner is:

$$\Delta x' = \gamma(\Delta x - u\Delta t).$$

We set $\Delta t = 0$ and solve for Δx:

$$\Delta x = \frac{1}{\gamma}\Delta x'.$$

We see that the ship's length measured by Jocelyn is proportional to $1/\gamma$. Since $\gamma \geq 1$, then $\Delta x \leq \Delta x'$. In other words, the length in the x direction will be *contracted* relative to the length measured when the ship is at rest.

If we call the contracted length of the ship $L = \Delta x$, then we can write the above equation as

$$L = \frac{L_0}{\gamma} = \sqrt{1 - u^2/c^2}\,L_0. \tag{2.2}$$

2.6 Length and Time Interval Measurements

The **proper time interval** τ_0 measures the time between two events that take place at the same location in space according to the observer. The time-dilated interval measures the time between two events that occur at different locations.

The **proper length** L_0 of an object measures the distance between the ends of the object when it is at rest with respect to the observer. Another way of saying this is that it measures the distance between events that occur simultaneously according to the observer.

The relationships between velocity u, relative time τ, proper time τ_0, relative distance L and proper distance L_0 may be summarized as follows:

$$u = \frac{L_0}{\tau} \qquad u = \frac{L}{\tau_0}. \tag{2.3}$$

We also observe that

$$\frac{\tau}{\tau_0} = \frac{L_0}{L} = \gamma.$$

2.7 Low Velocity Limit

When the relative velocity between two frames of reference is much less than c, we must sometimes use the binomial expansion to approximate the Lorentz factor γ. For example, the Lorentz factor γ for a speed of 10 m/s is

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1 - (10/3 \times 10^8)^2}} \approx 1$$

when you type it into your calculator. The point is that it is *very* close to, but not exactly equal to unity. To calculate it, use the binomial series expansion:

$$(1 - x)^n \approx 1 - nx.$$

Applying this to the Lorentz factor gives

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} \tag{2.4}$$

for $u << c$. When $u = 10$ m/s, we can now evaluate

$$\gamma \approx 1 + \frac{1}{2}\frac{10^2}{(3 \times 10^8)^2} = 1 + 5.56 \times 10^{-16}$$

or

$$\gamma \approx 1.000000000000000556.$$

2.8 Trip to a Star

Suppose Vera is on the Earth and she measures the distance to the star Vega to be 25 light years. One light year (l.y.) equals the distance light travels in one year. Jocelyn is flying by Earth in a spaceship traveling at a relative speed of $u = 0.98c$. How long will it take Jocelyn to reach Vega?

Answer the question from both Vera and Jocelyn's perspectives.

Vera's Perspective. Because Vera is at rest with respect to the Earth and Vega, she measures their proper distance. So

$$L_0 = 25 \text{ l.y.}$$

Because she sees Jocelyn moving at velocity u, she will see her reach Vega in a time

$$\tau = \frac{L_0}{u} = \frac{25 \text{ l.y.}}{0.98c} = 25.51 \text{ l.y.}/c = 25.51 \text{ years.}$$

Note: it is impossible for her to see Jocelyn reach Vega in less than 25 years because that would imply faster-than-light travel, which relativity prohibits. The time that Vera measures is *not* the proper time because the events (Jocelyn passing by Earth and Jocelyn arriving at Vega) did not occur at the same spatial position in Vera's frame.

Jocelyn's Perspective. She is moving relative to the Earth and Vega. She will see them both approaching her at a speed $0.98c$. Thus, she will see the distance between them as being length contracted: $L = L_0/\gamma$ where $\gamma = 1/\sqrt{1-(0.98)^2} = 5.03$. She sees the distance between Earth and Vega as

$$L = (25 \text{ l.y.})/5.03 = 4.97 \text{ l.y.}$$

Because both events (her passing Earth and her arrival at Vega) occur at the same spatial coordinate in her frame, she will measure the proper time:

$$\tau_0 = \frac{\tau}{\gamma} = \frac{25.52 \text{ years}}{5.03} = 5.07 \text{ years.}$$

Notice that Jocelyn will measure the same relative speed between her and Earth as Vera did:

$$u = \frac{L}{\tau_0} = \frac{4.97 \text{ l.y.}}{5.07 \text{ years}} = 0.98c$$